ALL IN ONE

MATHEMATICS CHEAT SHEET

V2.10

Euler's Identity:

$$e^{i\pi} + 1 = 0$$

CONTAINING FORMULAE FOR ELEMENTARY, HIGH SCHOOL AND UNIVERSITY MATHEMATICS

COMPILED FROM MANY SOURCES BY ALEX SPARTALIS

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UPDATED: Composite Functions

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NEW: Three Phase – Delta & Y
NEW: Electrical Power

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NEW: Factorial
NEW: Electromagnetics
NEW: Linear Algebra
NEW: Mathematical Symbols
NEW: Algebraic Identities
NEW: Graph Theory
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NEW: Graphical Functions
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NEW: Machin-Like Formulae
NEW: Infinite Summations To Pi
NEW: Classical Mechanics
NEW: Relativistic Formulae
NEW: Statistical Distributions
NEW: Logarithm Power Series
NEW: Spherical Triangle Identities
NEW: Bernoulli Expansion
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UPDATED: Riemann Zeta Function
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NEW: QR Factorisation
NEW: Jordan Forms
NEW: Macroeconomics
NEW: Golden Ratio & Fibonacci Sequence
NEW: Complex Vectors and Matrices
NEW: Numerical Computations for Matrices
UPDATED: Prime Numbers
UPDATED: Errors within Matrix Formula

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NEW: USV Decomposition
NEW: Ordinary Differential Equations Using Matrices
NEW: Exponential Identities
UPDATED: Matrix Inverse
CORRECTION: Left and Right Matrix Inverse

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NEW: Applications of Functions
NEW: Higher Order Integration
NEW: Root Expansions

NEW: Mathematical Constants
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UPDATED: Basic Statistical Operations
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UPDATED: Basic Statistical Operations:
UPDATED: Discrete Random Variables:
UPDATED: Common DRVs:
UPDATED: Undetermined Coefficients
UPDATED: Variation of Parameters

CONTENTS

PART 1: PHYSICAL CONSTANTS

1.1 SI PREFIXES:

Prefix	Symbol	1000^m	10^n	Decimal	Scale
yotta	Y	1000^8	10^{24}	1000000000000000000000000	Septillion
zetta	Z	1000^7	10^{21}	1000000000000000000000	Sextillion
exa	E	1000^6	10^{18}	1000000000000000000	Quintillion
peta	P	1000^5	10^{15}	1000000000000000	Quadrillion
tera	T	1000^4	10^{12}	1000000000000	Trillion
giga	G	1000^3	10^9	1000000000	Billion
mega	M	1000^2	10^6	1000000	Million
kilo	k	1000^1	10^3	1000	Thousand
hecto	h	$1000^{2/3}$	10^2	100	Hundred
deca	da	$1000^{1/3}$	10^1	10	Ten
		1000^0	10^0	1	One
deci	d	$1000^{-1/3}$	10^{-1}	0.1	Tenth
centi	c	$1000^{-2/3}$	10^{-2}	0.01	Hundredth
milli	m	1000^{-1}	10^{-3}	0.001	Thousandth
micro	µ	1000^{-2}	10^{-6}	0.000001	Millionth
nano	n	1000^{-3}	10^{-9}	0.000000001	Billionth
pico	p	1000^{-4}	10^{-12}	0.000000000001	Trillionth
femto	f	1000^{-5}	10^{-15}	0.000000000000001	Quadrillionth
atto	a	1000^{-6}	10^{-18}	0.000000000000000001	Quintillionth
zepto	z	1000^{-7}	10^{-21}	0.000000000000000000001	Sextillionth
yocto	y	1000^{-8}	10^{-24}	0.000000000000000000000001	Septillionth

1.2 SI BASE UNITS:

Quantity	Unit	Symbol
length	meter	m
mass	kilogram	kg
time	second	s
electric current	ampere	A
thermodynamic temperature	kelvin	K
amount of substance	mole	mol
luminous intensity	candela	cd

1.3 SI DERIVED UNITS:

Quantity	Unit	Symbol	Expression in terms of other SI units
angle, plane	radian*	rad	m/m = 1
angle, solid	steradian*	sr	$m^2/m^2 = 1$
Celsius temperature	degree Celsius	°C	K
electric capacitance	farad	F	C/V
electric charge, quantity of electricity	coulomb	C	A·s
electric conductance	siemens	S	A/V
electric inductance	henry	H	Wb/A
electric potential difference, electromotive force	volt	V	W/A
electric resistance	ohm	Ω	V/A
energy, work, quantity of heat	joule	J	N·m
force	newton	N	$kg·m/s^2$
frequency (of a periodic phenomenon)	hertz	Hz	1/s
illuminance	lux	lx	lm/m^2
luminous flux	lumen	lm	cd·sr
magnetic flux	weber	Wb	V·s
magnetic flux density	tesla	T	Wb/m^2
power, radiant flux	watt	W	J/s
pressure, stress	pascal	Pa	N/m^2
activity (referred to a radionuclide)	becquerel	Bq	1/s
absorbed dose, specific energy imparted, kerma	gray	Gy	J/kg
dose equivalent, ambient dose equivalent, directional dose equivalent, personal dose equivalent, organ dose equivalent	sievert	Sv	J/kg
catalytic activity	katal	kat	mol/s

1.4 UNIVERSAL CONSTANTS:

Quantity	Symbol	Value	Relative Standard Uncertainty
speed of light in vacuum	c	299 792 458 m·s^{-1}	defined
Newtonian constant of gravitation	G	6.67428(67)×10^{-11} m^{3}·kg^{-1}·s^{-2}	1.0 × 10^{-4}
Planck constant	h	6.626 068 96(33) × 10^{-34} J·s	5.0 × 10^{-8}
reduced Planck constant	$\hbar = h/(2\pi)$	1.054 571 628(53) × 10^{-34} J·s	5.0 × 10^{-8}

1.5 ELECTROMAGNETIC CONSTANTS:

Quantity	Symbol	Value (SI units)	Relative Standard Uncertainty
magnetic constant (vacuum permeability)	μ_0	4π × 10^{-7} N·A^{-2} = 1.256 637 061... × 10^{-6} N·A^{-2}	defined
electric constant (vacuum permittivity)	$\varepsilon_0 = 1/(\mu_0 c^2)$	8.854 187 817... × 10^{-12} F·m^{-1}	defined
characteristic impedance of vacuum	$Z_0 = \mu_0 c$	376.730 313 461... Ω	defined
Coulomb's constant	$k_e = 1/4\pi\epsilon_0$	8.987 551 787... × 10^{9} N·m²·C^{-2}	defined
elementary charge	e	1.602 176 487(40) × 10^{-19} C	2.5 × 10^{-8}
Bohr magneton	$\mu_B = e\hbar/2m_e$	927.400 915(23) × 10^{-26} J·T^{-1}	2.5 × 10^{-8}
conductance quantum	$G_0 = 2e^2/h$	7.748 091 7004(53) × 10^{-5} S	6.8 × 10^{-10}
inverse conductance quantum	$G_0^{-1} = h/2e^2$	12 906.403 7787(88) Ω	6.8 × 10^{-10}
Josephson constant	$K_J = 2e/h$	4.835 978 91(12) × 10^{14} Hz·V^{-1}	2.5 × 10^{-8}
magnetic flux quantum	$\phi_0 = h/2e$	2.067 833 667(52) × 10^{-15} Wb	2.5 × 10^{-8}
nuclear magneton	$\mu_N = e\hbar/2m_p$	5.050 783 43(43) × 10^{-27} J·T^{-1}	8.6 × 10^{-8}
von Klitzing constant	$R_K = h/e^2$	25 812.807 557(18) Ω	6.8 × 10^{-10}

1.6 ATOMIC AND NUCLEAR CONSTANTS:

Quantity	Symbol	Value (SI units)	Relative Standard Uncertainty
Bohr radius	$a_0 = \alpha/4\pi R_\infty$	5.291 772 108(18) × 10^{-11} m	3.3 × 10^{-9}
classical electron radius	$r_e = e^2/4\pi\epsilon_0 m_e c^2$	2.817 940 2894(58) × 10^{-15} m	2.1 × 10^{-9}
electron mass	m_e	9.109 382 15(45) × 10^{-31} kg	5.0 × 10^{-8}
Fermi coupling constant	$G_F/(\hbar c)^3$	1.166 39(1) × 10^{-5} GeV^{-2}	8.6 × 10^{-6}
fine-structure constant	$\alpha = \mu_0 e^2 c/(2h) = e^2/(4\pi\epsilon_0\hbar c)$	7.297 352 537 6(50) × 10^{-3}	6.8 × 10^{-10}
Hartree energy	$E_h = 2R_\infty hc$	4.359 744 17(75) × 10^{-18} J	1.7 × 10^{-7}
proton mass	m_p	1.672 621 637(83) × 10^{-27} kg	5.0 × 10^{-8}
quantum of circulation	$h/2m_e$	3.636 947 550(24) × 10^{-4} m^2 s^{-1}	6.7 × 10^{-9}
Rydberg constant	$R_\infty = \alpha^2 m_e c/2h$	10 973 731.568 525(73) m^{-1}	6.6 × 10^{-12}
Thomson cross section	$(8\pi/3)r_e^2$	6.652 458 73(13) × 10^{-29} m^2	2.0 × 10^{-8}
weak mixing angle	$\sin^2\theta_W = 1 - (m_W/m_Z)^2$	0.222 15(76)	3.4 × 10^{-3}

1.7 PHYSICO-CHEMICAL CONSTANTS:

Quantity		Symbol	Value (SI units)	Relative Standard Uncertainty
atomic mass unit (unified atomic mass unit)		$m_u = 1\ u$	1.660 538 86(28) × 10^{-27} kg	1.7 × 10^{-7}
Avogadro's number		N_A, L	6.022 141 5(10) × 10^{23} mol^{-1}	1.7 × 10^{-7}
Boltzmann constant		$k = k_B = R/N_A$	1.380 6504(24) × 10^{-23} $J\cdot K^{-1}$	1.8 × 10^{-6}
Faraday constant		$F = N_A e$	96 485.3383(83)$C\cdot mol^{-1}$	8.6 × 10^{-8}
first radiation		$c_1 = 2\pi hc^2$	3.741 771 18(19) × 10^{-16} $W\cdot m^2$	5.0 × 10^{-8}

constant	for spectral radiance	c_{1L}	1.191 042 82(20) × 10^{-16} W·m² sr^{-1}	1.7 × 10^{-7}
Loschmidt constant	at T=273.15 K and p=101.325 kPa	$n_0 = N_A/V_m$	2.686 777 3(47) × 10^{25} m^{-3}	1.8 × 10^{-6}
gas constant		R	8.314 472(15) J·K^{-1}·mol^{-1}	1.7 × 10^{-6}
molar Planck constant		$N_A h$	3.990 312 716(27) × 10^{-10} J·s·mol^{-1}	6.7 × 10^{-9}
molar volume of an ideal gas	at T=273.15 K and p=100 kPa	$V_m = RT/p$	2.2710 981(40) × 10^{-2} m³·mol^{-1}	1.7 × 10^{-6}
molar volume of an ideal gas	at T=273.15 K and p=101.325 kPa	$V_m = RT/p$	2.2413 996(39) × 10^{-2} m³·mol^{-1}	1.7 × 10^{-6}
Sackur-Tetrode constant	at T=1 K and p=100 kPa	$S_0/R = \frac{5}{2} + \ln\left[(2\pi m_u kT/h^2)^{3/2} kT/p\right]$	−1.151 704 7(44)	3.8 × 10^{-6}
Sackur-Tetrode constant	at T=1 K and p=101.325 kPa	$S_0/R = \frac{5}{2} + \ln\left[(2\pi m_u kT/h^2)^{3/2} kT/p\right]$	−1.164 867 7(44)	3.8 × 10^{-6}
second radiation constant		$c_2 = hc/k$	1.438 775 2(25) × 10^{-2} m·K	1.7 × 10^{-6}
Stefan–Boltzmann constant		$\sigma = (\pi^2/60)k^4/\hbar^3 c^2$	5.670 400(40) × 10^{-8} W·m^{-2}·K^{-4}	7.0 × 10^{-6}
Wien displacement law constant		$b = (hc/k)/$4.965 114 231...	2.897 768 5(51) × 10^{-3} m·K	1.7 × 10^{-6}

1.8 ADOPTED VALUES:

Quantity		Symbol	Value (SI units)	Relative Standard Uncertainty
conventional value of Josephson constant		K_{J-90}	4.835 979 × 10^{14} Hz·V^{-1}	defined
conventional value of von Klitzing constant		R_{K-90}	25 812.807 Ω	defined
molar mass	constant	$M_u = M(^{12}\mathrm{C})/12$	1 × 10^{-3} kg·mol^{-1}	defined
molar mass	of carbon-12	$M(^{12}\mathrm{C}) = N_A m(^{12}\mathrm{C})$	1.2 × 10^{-2}	defined

		$\mathrm{kg \cdot mol^{-1}}$	
standard acceleration of gravity (gee, free-fall on Earth)	g_n	9.806 65 $\mathrm{m \cdot s^{-2}}$	defined
standard atmosphere	atm	101 325 Pa	defined

1.9 NATURAL UNITS:

Name	Dimension	Expression	Value (SI units)
Planck length	Length (L)	$l_P = \sqrt{\frac{\hbar G}{c^3}}$	1.616 252(81) × 10^{-35} m
Planck mass	Mass (M)	$m_P = \sqrt{\frac{\hbar c}{G}}$	2.176 44(11) × 10^{-8} kg
Planck time	Time (T)	$t_P = \sqrt{\frac{\hbar G}{c^5}}$	5.391 24(27) × 10^{-44} s
Planck charge	Electric charge (Q)	$q_P = \sqrt{4\pi\varepsilon_0 \hbar c}$	1.875 545 870(47) × 10^{-18} C
Planck temperature	Temperature (Θ)	$T_P = \sqrt{\frac{\hbar c^5}{G k^2}}$	1.416 785(71) × 10^{32} K

1.10 MATHEMATICAL CONSTANTS:

(each to 1000 decimal places)

π ≈
3.14159265358979323846264338327950288419716939937510582097494459230781640628620899862803482
5342117067982148086513282306647093844609550582231725359408128481117450284102701938521105559
6446229489549303819644288109756659334461284756482337867831652712019091456485669234603486104
5432664821339360726024914127372458700660631558817488152092096282925409171536436789259036001
1330530548820466521384146951941511609433057270365759591953092186117381932611793105118548074
4623799627495673518857527248912279381830119491298336733624406566430860213949463952247371907
0217986094370277053921717629317675238467481846766940513200056812714526356082778577134275778
9609173637178721468440901224953430146549585371050792279689258923542019956112129021960864034
4181598136297747713099605187072113499999983729780499510597317328160963185950244594553469083
0264252230825334468503526193118817101000313783875288658753320838142061717766914730359825349
0428755468731159562863882353787593751957781857780532171226806613001927876611195909216420199

e ≈
2.71828182845904523536028747135266249775724709369995957496696762772407663035354759457138217
8525166427427466391932003059921817413596629043572900334295260595630738132328627943490763233
8298807531952510190115738341879307021540891499348841675092447614606680822648001684774118537
4234544243710753907774499206955170276183860626133138458300075204493382656029760673711320070
9328709127443747047230696977209310141692836819025515108657463772111252389784425056953696770
7854499699679468644549059879316368892300987931277361782154249992295763514822082698951936680
3318252886939849646510582093923982948879332036250944311730123819706841614039701983767932068
3282376464804295311802328782509819455815301756717361332069811250996181881593041690351598888
5193458072738667385894228792284998920868058257492796104841984443634632449684875602336248270
4197862320900216099023530436994184914631409343173814364054625315209618369088870701676839642
4378140592714563549061303107208510383750510115747704171898610687396965521267154688957035035

φ ≈

1.6180339887498948482045868343656381177203091798057628621354486227052604628189024497072072041893911374847540880753868917521266338622235369317931800607667263544333890865959395829056383226613199282902678806752087668925017116962070322210432162695486262963136144381497587012203408058879544547492461856953648644492410443207713449470495658467885098743394422125448770664780915884607499887124007652170575179788341662562494075890697040002812104276217711177780531531714101170466659914669798731761356006708748071013179523689427521948435305678300228785699782977834784587822891109762500302696156170025046433824377648610283831268330372429267526311653392473167111211588186385133162038400522216579128667529465490681131715993432359734949850904094762132229810172610705961164562990981629055520852479035240602017279974717534277759277862561943208275051312181562855122248093947123414517022373580577278616008688382952304592647878017889921990270776903895321968198615143780314997411069260886742962267575605231727775203536139362

PART 2: MATHEMTAICAL SYMBOLS

2.1 BASIC MATH SYMBOLS

Symbol	Symbol Name	Meaning / definition	Example
=	equals sign	equality	5 = 2+3
≠	not equal sign	inequality	5 ≠ 4
>	strict inequality	greater than	5 > 4
<	strict inequality	less than	4 < 5
≥	inequality	greater than or equal to	5 ≥ 4
≤	inequality	less than or equal to	4 ≤ 5
( )	parentheses	calculate expression inside first	2 × (3+5) = 16
[ ]	brackets	calculate expression inside first	[(1+2)*(1+5)] = 18
+	plus sign	addition	1 + 1 = 2
−	minus sign	subtraction	2 − 1 = 1
±	plus - minus	both plus and minus operations	3 ± 5 = 8 and -2
∓	minus - plus	both minus and plus operations	3 ∓ 5 = -2 and 8
*	asterisk	multiplication	2 * 3 = 6
×	times sign	multiplication	2 × 3 = 6
·	multiplication dot	multiplication	2 · 3 = 6
÷	division sign / obelus	division	6 ÷ 2 = 3
/	division slash	division	6 / 2 = 3
–	horizontal line	division / fraction	$\frac{6}{2} = 3$
mod	modulo	remainder calculation	7 mod 2 = 1
.	period	decimal point, decimal separator	2.56 = 2+56/100
a^b	power	exponent	$2^3 = 8$
a^*b*	caret	exponent	2 ^ 3 = 8
$\sqrt{a}$	square root	$\sqrt{a} \cdot \sqrt{a} = a$	$\sqrt{9} = \pm 3$
$\sqrt[3]{a}$	cube root		$\sqrt[3]{8} = 2$
$\sqrt[4]{a}$	forth root		$\sqrt[4]{16} = \pm 2$
$\sqrt[n]{a}$	n-th root (radical)		for n=3, $\sqrt[n]{8} = 2$
%	percent	1% = 1/100	10% × 30 = 3
‰	per-mille	1‰ = 1/1000 = 0.1%	10‰ × 30 = 0.3
ppm	per-million	1ppm = 1/1000000	10ppm × 30 = 0.0003
ppb	per-billion	1ppb = 1/1000000000	10ppb × 30 = 3×10^{-7}
ppt	per-trillion	1ppb = 10^{-12}	10ppb × 30 = 3×10^{-10}

2.2 GEOMETRY SYMBOLS

Symbol	Symbol Name	Meaning / definition	Example
∠	angle	formed by two rays	∠ABC = 30º
∡	measured angle		∡ABC = 30º
∢	spherical angle		∢AOB = 30º
∟	right angle	= 90º	α = 90º
º	degree	1 turn = 360º	α = 60º
´	arcminute	1º = 60´	α = 60º59'

´´	arcsecond	1´ = 60´´	α = 60º59'59"
AB	line	line from point A to point B	
$\overrightarrow{AB}$	ray	line that start from point A	
⊥	perpendicular	perpendicular lines (90º angle)	AC ⊥ BC
||	parallel	parallel lines	AB || CD
≅	congruent to	equivalence of geometric shapes and size	∆ABC ≅ ∆XYZ
~	similarity	same shapes, not same size	∆ABC ~ ∆XYZ
Δ	triangle	triangle shape	ΔABC ≅ ΔBCD
$\|x-y\|$	distance	distance between points x and y	$\|x-y\| = 5$
π	pi constant	$\pi = 3.141592654...$ is the ratio between the circumference and diameter of a circle	$c = \pi \cdot d = 2 \cdot \pi \cdot r$
rad	radians	radians angle unit	360º = 2π rad
grad	grads	grads angle unit	360º = 400 grad

2.3 ALGEBRA SYMBOLS

Symbol	Symbol Name	Meaning / definition	Example
x	x variable	unknown value to find	when $2x = 4$, then $x = 2$
≡	equivalence	identical to	
≜	equal by definition	equal by definition	
:=	equal by definition	equal by definition	
~	approximately equal	weak approximation	11 ~ 10
≈	approximately equal	approximation	$sin(0.01) \approx 0.01$
∝	proportional to	proportional to	$f(x) \propto g(x)$
∞	lemniscate	infinity symbol	
≪	much less than	much less than	1 ≪ 1000000
≫	much greater than	much greater than	1000000 ≫ 1
( )	parentheses	calculate expression inside first	2 * (3+5) = 16
[ ]	brackets	calculate expression inside first	[(1+2)*(1+5)] = 18
{ }	braces	set	
$\lfloor x \rfloor$	floor brackets	rounds number to lower integer	$\lfloor 4.3 \rfloor = 4$
$\lceil x \rceil$	ceiling brackets	rounds number to upper integer	$\lceil 4.3 \rceil = 5$
$x!$	exclamation mark	factorial	4! = 1*2*3*4 = 24
$\|x\|$	single vertical bar	absolute value	| -5 | = 5
$f(x)$	function of x	maps values of x to f(x)	$f(x) = 3x+5$
$(f \circ g)$	function composition	$(f \circ g)(x) = f(g(x))$	$f(x)=3x, g(x)=x-1 \Rightarrow (f \circ g)(x)=3(x-1)$
(a,b)	open interval	$(a,b) \triangleq \{x \mid a < x < b\}$	$x \in (2,6)$
$[a,b]$	closed interval	$[a,b] \triangleq \{x \mid a \le x \le b\}$	$x \in [2,6]$
∆	delta	change / difference	$\Delta t = t_1 - t_0$
∆	discriminant	$\Delta = b^2 - 4ac$	
∑	sigma	summation - sum of all values in range of series	$\sum x_i = x_1+x_2+...+x_n$

∑∑	sigma	double summation	$\sum_{j=1}^{2}\sum_{i=1}^{8} x_{i,j} = \sum_{i=1}^{8} x_{i,1} + \sum_{i=1}^{8} x_{i,2}$
∏	capital pi	product - product of all values in range of series	$\prod x_i = x_1 \cdot x_2 \cdot \ldots \cdot x_n$
e	e constant / Euler's number	*e* = 2.718281828...	$e = \lim (1+1/x)^x$, $x \to \infty$
γ	Euler-Mascheroni constant	γ = 0.527721566...	
φ	golden ratio	golden ratio constant	

2.4 LINEAR ALGEBRA SYMBOLS

Symbol	Symbol Name	Meaning / definition	Example
·	dot	scalar product	$a \cdot b$
×	cross	vector product	$a \times b$
$A \otimes B$	tensor product	tensor product of A and B	$A \otimes B$
$\langle x, y \rangle$	inner product		
[ ]	brackets	matrix of numbers	
( )	parentheses	matrix of numbers	
$\| A \|$	determinant	determinant of matrix A	
det(A)	determinant	determinant of matrix A	
$\|\| x \|\|$	double vertical bars	norm	
A^T	transpose	matrix transpose	$(A^T)_{ij} = (A)_{ji}$
$A^\dagger$	Hermitian matrix	matrix conjugate transpose	$(A^\dagger)_{ij} = (A)_{ji}$
A^*	Hermitian matrix	matrix conjugate transpose	$(A^*)_{ij} = (A)_{ji}$
A^{-1}	inverse matrix	$A A^{-1} = I$	
rank(A)	matrix rank	rank of matrix A	rank(A) = 3
dim(U)	dimension	dimension of matrix A	rank(U) = 3

2.5 PROBABILITY AND STATISTICS SYMBOLS

Symbol	Symbol Name	Meaning / definition	Example
$P(A)$	probability function	probability of event A	$P(A) = 0.5$
$P(A \cap B)$	probability of events intersection	probability that of events A and B	$P(A \cap B) = 0.5$
$P(A \cup B)$	probability of events union	probability that of events A or B	$P(A \cup B) = 0.5$
$P(A \mid B)$	conditional probability function	probability of event A given event B occured	$P(A \mid B) = 0.3$
$f(x)$	probability density function (pdf)	$P(a \le x \le b) = \int f(x)\, dx$	
$F(x)$	cumulative distribution function (cdf)	$F(x) = P(X \le x)$	
μ	population mean	mean of population values	$\mu = 10$
$E(X)$	expectation value	expected value of random variable X	$E(X) = 10$
$E(X \mid Y)$	conditional expectation	expected value of random variable X given Y	$E(X \mid Y=2) = 5$

$var(X)$	variance	variance of random variable X	$var(X) = 4$
σ^2	variance	variance of population values	$\sigma^2 = 4$
$std(X)$	standard deviation	standard deviation of random variable X	$std(X) = 2$
σ_X	standard deviation	standard deviation value of random variable X	$\sigma_X = 2$
$\tilde{x}$	median	middle value of random variable x	$\tilde{x} = 5$
$cov(X,Y)$	covariance	covariance of random variables X and Y	$cov(X,Y) = 4$
$corr(X,Y)$	correlation	correlation of random variables X and Y	$corr(X,Y) = 3$
$\rho_{X,Y}$	correlation	correlation of random variables X and Y	$\rho_{X,Y} = 3$
$\sum$	summation	summation - sum of all values in range of series	$\sum_{i=1}^{4} x_i = x_1 + x_2 + x_3 + x_4$
$\sum\sum$	double summation	double summation	$\sum_{j=1}^{2} \sum_{i=1}^{8} x_{i,j} = \sum_{i=1}^{8} x_{i,1} + \sum_{i=1}^{8} x_{i,2}$
Mo	mode	value that occurs most frequently in population	
MR	mid-range	$MR = (x_{max}+x_{min})/2$	
Md	sample median	half the population is below this value	
Q_1	lower / first quartile	25% of population are below this value	
Q_2	median / second quartile	50% of population are below this value = median of samples	
Q_3	upper / third quartile	75% of population are below this value	
x	sample mean	average / arithmetic mean	$x = (2+5+9) / 3 = 5.333$
s^2	sample variance	population samples variance estimator	$s^2 = 4$
s	sample standard deviation	population samples standard deviation estimator	$s = 2$
z_x	standard score	$z_x = (x-x) / s_x$	
$X \sim$	distribution of X	distribution of random variable X	$X \sim N(0,3)$
$N(\mu,\sigma^2)$	normal distribution	gaussian distribution	$X \sim N(0,3)$
$U(a,b)$	uniform distribution	equal probability in range a,b	$X \sim U(0,3)$
$exp(\lambda)$	exponential distribution	$f(x) = \lambda e^{-\lambda x}, x \geq 0$	
$gamma(c, \lambda)$	gamma distribution	$f(x) = \lambda c x^{c-1} e^{-\lambda x} / \Gamma(c), x \geq 0$	
$\chi^2(k)$	chi-square distribution	$f(x) = x^{k/2-1} e^{-x/2} / (2^{k/2} \Gamma(k/2))$	
$F(k_1, k_2)$	F distribution		
$Bin(n,p)$	binomial distribution	$f(k) = {}_nC_k\, p^k (1-p)^{n-k}$	
$Poisson(\lambda)$	Poisson distribution	$f(k) = \lambda^k e^{-\lambda} / k!$	
$Geom(p)$	geometric distribution	$f(k) = p(1-p)^k$	
$HG(N,K,n)$	hyper-geometric distribution		
$Bern(p)$	Bernoulli distribution		

2.6 COMBINATORICS SYMBOLS

Symbol	Symbol Name	Meaning / definition	Example
$n!$	factorial	$n! = 1\cdot 2\cdot 3\cdot ...\cdot n$	$5! = 1\cdot 2\cdot 3\cdot 4\cdot 5 = 120$
$_nP_k$	permutation	$_nP_k = \frac{n!}{(n-k)!}$	$_5P_3 = 5! / (5-3)! = 60$
$_nC_k$ $\binom{n}{k}$	combination	$_nC_k = \binom{n}{k} = \frac{n!}{k!(n-k)!}$	$_5C_3 = 5!/[3!(5-3)!]=10$

2.7 SET THEORY SYMBOLS

Symbol	Symbol Name	Meaning / definition	Example
{ }	set	a collection of elements	A={3,7,9,14}, B={9,14,28}
A ∩ B	intersection	objects that belong to set A and set B	A ∩ B = {9,14}
A ∪ B	union	objects that belong to set A or set B	A ∪ B = {3,7,9,14,28}
A ⊆ B	subset	subset has less elements or equal to the set	{9,14,28} ⊆ {9,14,28}
A ⊂ B	proper subset / strict subset	subset has less elements than the set	{9,14} ⊂ {9,14,28}
A ⊄ B	not subset	left set not a subset of right set	{9,66} ⊄ {9,14,28}
A ⊇ B	superset	set A has more elements or equal to the set B	{9,14,28} ⊇ {9,14,28}
A ⊃ B	proper superset / strict superset	set A has more elements than set B	{9,14,28} ⊃ {9,14}
A ⊅ B	not superset	set A is not a superset of set B	{9,14,28} ⊅ {9,66}
2^A	power set	all subsets of A	
$\mathcal{P}(A)$	power set	all subsets of A	
A = B	equality	both sets have the same members	A={3,9,14}, B={3,9,14}, A=B
A^c	complement	all the objects that do not belong to set A	
A \ B	relative complement	objects that belong to A and not to B	A={3,9,14}, B={1,2,3}, A-B={9,14}
A - B	relative complement	objects that belong to A and not to B	A={3,9,14}, B={1,2,3}, A-B={9,14}
A ∆ B	symmetric difference	objects that belong to A or B but not to their intersection	A={3,9,14}, B={1,2,3}, A ∆ B={1,2,9,14}
A ⊖ B	symmetric difference	objects that belong to A or B but not to their intersection	A={3,9,14}, B={1,2,3}, A ⊖ B={1,2,9,14}
a∈A	element of	set membership	A={3,9,14}, 3 ∈ A
x∉A	not element of	no set membership	A={3,9,14}, 1 ∉ A
(a,b)	ordered pair	collection of 2 elements	
A×B	cartesian product	set of all ordered pairs from A and B	
|A|	cardinality	the number of elements of set A	A={3,9,14}, |A|=3
#A	cardinality	the number of elements of set A	A={3,9,14}, #A=3

ℵ	aleph	infinite cardinality	
Ø	empty set	Ø = { }	C = {Ø}
U	universal set	set of all possible values	
$\mathbb{N}_0$	natural numbers set (with zero)	$\mathbb{N}_0 = \{0,1,2,3,4,...\}$	$0 \in \mathbb{N}_0$
$\mathbb{N}_1$	natural numbers set (without zero)	$\mathbb{N}_1 = \{1,2,3,4,5,...\}$	$6 \in \mathbb{N}_1$
$\mathbb{Z}$	integer numbers set	$\mathbb{Z} = \{...-3,-2,-1,0,1,2,3,...\}$	$-6 \in \mathbb{Z}$
$\mathbb{Q}$	rational numbers set	$\mathbb{Q} = \{x \mid x=a/b, a,b\in\mathbb{N}\}$	$2/6 \in \mathbb{Q}$
$\mathbb{R}$	real numbers set	$\mathbb{R} = \{x \mid -\infty < x < \infty\}$	$6.343434 \in \mathbb{R}$
$\mathbb{C}$	complex numbers set	$\mathbb{C} = \{z \mid z=a+bi, -\infty<a<\infty, -\infty<b<\infty\}$	$6+2i \in \mathbb{C}$

2.8 LOGIC SYMBOLS

Symbol	Symbol Name	Meaning / definition	Example
·	and	and	$x \cdot y$
^	caret / circumflex	and	x ^ y
&	ampersand	and	x & y
+	plus	or	$x + y$
∨	reversed caret	or	$x \vee y$
|	vertical line	or	$x \mid y$
x'	single quote	not - negation	x'
x	bar	not - negation	x
¬	not	not - negation	$\neg x$
!	exclamation mark	not - negation	$! x$
⊕	circled plus / oplus	exclusive or - xor	$x \oplus y$
~	tilde	negation	$\sim x$
⇒	implies		
⇔	equivalent	if and only if	
∀	for all		
∃	there exists		
∄	there does not exists		
∴	therefore		
∵	because / since		

2.9 CALCULUS & ANALYSIS SYMBOLS

Symbol	Symbol Name	Meaning / definition	Example
$\lim_{x\to x0} f(x)$	limit	limit value of a function	
ε	epsilon	represents a very small number, near zero	$\varepsilon \to 0$
e	e constant / Euler's number	e = 2.718281828...	$e = \lim (1+1/x)^x$, $x\to\infty$
y'	derivative	derivative - Leibniz's notation	$(3x^3)' = 9x^2$
y''	second derivative	derivative of derivative	$(3x^3)'' = 18x$
$y^{(n)}$	nth derivative	n times derivation	$(3x^3)^{(3)} = 18$
$\frac{dy}{dx}$	derivative	derivative - Lagrange's notation	$d(3x^3)/dx = 9x^2$
$\frac{d^2y}{dx^2}$	second derivative	derivative of derivative	$d^2(3x^3)/dx^2 = 18x$
$\frac{d^ny}{dx^n}$	nth derivative	n times derivation	
$\dot{y}$	time derivative	derivative by time - Newton notation	
$\ddot{y}$	time second derivative	derivative of derivative	
$\frac{\partial f(x,y)}{\partial x}$	partial derivative		$\partial(x^2+y^2)/\partial x = 2x$
$\int$	integral	opposite to derivation	
$\iint$	double integral	integration of function of 2 variables	
$\iiint$	triple integral	integration of function of 3 variables	
$\oint$	closed contour / line integral		
$\oiint$	closed surface integral		
$\oiiint$	closed volume integral		
$[a,b]$	closed interval	$[a,b] = \{x \mid a \le x \le b\}$	
(a,b)	open interval	$(a,b) = \{x \mid a < x < b\}$	
i	imaginary unit	$i \equiv \sqrt{-1}$	$z = 3 + 2i$
z^*	complex conjugate	$z = a+bi \to z^*=a-bi$	$z^* = 3 + 2i$
z	complex conjugate	$z = a+bi \to z = a-bi$	$z = 3 + 2i$
∇	nabla / del	gradient / divergence operator	$\nabla f(x,y,z)$
$\overrightarrow{x}$	vector		
$\widehat{x}$	unit vector		
$x * y$	convolution	$y(t) = x(t) * h(t)$	
$\mathcal{L}$	Laplace transform	$F(s) = \mathcal{L}\{f(t)\}$	
$\mathcal{F}$	Fourier transform	$X(\omega) = \mathcal{F}\{f(t)\}$	
δ	delta function		

PART 3: AREA, VOLUME AND SURFACE AREA

3.1 AREA:

Triangle: $A = \frac{1}{2}bh = \frac{1}{2}ab\sin C = \frac{a^2 \sin B \sin C}{2\sin A} = \sqrt{s(s-a)(s-b)(s-c)}$

Rectangle: $A = lw$

Square: $A = a^2$

Parallelogram: $A = bh = ab\sin A$

Rhombus: $A = a^2 \sin A$

Trapezium: $A = h\left(\frac{a+b}{s}\right)$

Quadrilateral: $A = \sqrt{(s-a)(s-b)(s-c)(s-d) - abcd \times \cos^2\left(\frac{\angle AB + \angle CD}{2}\right)}$

$A = \frac{d_1 d_2 \sin I}{2}$

Rectangle with rounded corners: $A = lw - r^2(4-\pi)$

Regular Hexagon: $A = \frac{3\sqrt{3} \times a^2}{2}$

Regular Octagon: $A = 2(1+\sqrt{2}) \times a^2$

Regular Polygon: $A = \frac{na^2}{4\tan\left(\frac{180}{n}\right)}$

3.2 VOLUME:

Cube: $V = a^3$

Cuboid: $V = abc$

Prisim: $V = A(b) \times h$

Pyramid: $V = \frac{1}{3} \times A(b) \times h$

Tetrahedron: $V = \frac{\sqrt{2}}{12} \times a^3$

Octahedron: $V = \frac{\sqrt{2}}{3} \times a^3$

Dodecahedron: $V = \frac{15+7\sqrt{5}}{4} \times a^3$

Icosahedron: $V = \frac{5(3+\sqrt{5})}{12} \times a^3$

3.3 SURFACE AREA:

Cube: $SA = 6a^2$

Cuboids: $SA = 2(ab + bc + ca)$

Tetrahedron: $SA = \sqrt{3} \times a^2$

Octahedron: $SA = 2 \times \sqrt{3} \times a^2$

Dodecahedron: $SA = 3 \times \sqrt{25 + 10\sqrt{5}} \times a^2$

Icosahedron: $SA = 5 \times \sqrt{3} \times a^2$

Cylinder: $SA = 2\pi r(h + r)$

3.4 MISCELLANEOUS:

Diagonal of a Rectangle $d = \sqrt{l^2 + w^2}$

Diagonal of a Cuboid $d = \sqrt{a^2 + b^2 + c^2}$

Longest Diagonal (Even Sides) $= \dfrac{a}{\sin\left(\dfrac{180}{n}\right)}$

Longest Diagonal (Odd Sides) $= \dfrac{a}{2\sin\left(\dfrac{90}{n}\right)}$

Total Length of Edges (Cube): $= 12a$

Total Length of Edges (Cuboid): $= 4(a + b + c)$

Circumference $C = 2\pi r = \pi d$

Perimeter of rectangle $P = 2(a + b)$

Semi perimeter $s = \dfrac{P}{2}$

Euler's Formula $Faces + Verticies = Edges + 2$

3.5 ABBREVIATIONS (3.1, 3.2, 3.3, 3.4)

A=area

a=side ‘a’

b=base

b=side ‘b’

C=circumference

C=central angle

c=side ‘c’

d=diameter

d=diagonal

d_1=diagonal 1

d_2=diagonal 2

E=external angle

h=height

I=internal angle

l=length

n=number of sides

P=perimeter

r=radius

r_1=radius 1
s=semi-perimeter
SA=Surface Area
V=Volume
w=width

PART 4: ALGEBRA & ARITHMETIC

4.1 POLYNOMIAL FORMULA:

Qudaratic: Where $ax^2 + bx + c = 0$,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Cubic: Where $ax^3 + bx^2 + cx + d = 0$,

Let, $x = y - \frac{b}{3a}$

$$\therefore a\left(y - \frac{b}{3a}\right)^3 + b\left(y - \frac{b}{3a}\right)^2 + c\left(y - \frac{b}{3a}\right) + d = 0$$

$$ay^3 + \left(c - \frac{b^2}{3a}\right)y + \left(d + \frac{2b^3}{27a^2} - \frac{bc}{3a}\right) = 0$$

$$y^3 + \frac{\left(c - \frac{b^2}{3a}\right)}{a}y + \frac{\left(d + \frac{2b^3}{27a^2} - \frac{bc}{3a}\right)}{a} = 0$$

$$y^3 + \frac{\left(c - \frac{b^2}{3a}\right)}{a}y = -\frac{\left(d + \frac{2b^3}{27a^2} - \frac{bc}{3a}\right)}{a}$$

Let, $A = \frac{\left(c - \frac{b^2}{3a}\right)}{a} = 3st \ldots (1)$

Let, $B = -\frac{\left(d + \frac{2b^3}{27a^2} - \frac{bc}{3a}\right)}{a} = s^3 - t^3 \ldots (2)$

$$\therefore y^3 + Ay = B$$

$$y^3 + 3sty = s^3 - t^3$$

Solution to the equation = $s - t$

Let, $y = s - t$

$$\therefore (s-t)^3 + 3st(s-t) = s^3 - t^3$$

$$\left(s^3 - 3s^2t + 3st^2 - t^3\right) + \left(3s^2t - 3st^2\right) = s^3 - t^3$$

Solving (1) for *s* and substituting into (2) yields:

$$\left(\frac{A}{3t}\right)^3 - t^3 = B.$$

$$t^6 + Bt^3 - \frac{A^3}{27} = 0,$$

Let, $u = t^3$

$$\therefore u^2 + Bu - \frac{A^3}{27} = 0$$

$$ie: \alpha u^2 + \beta u + \gamma = 0$$

$$\alpha = 1$$

$$\beta = B$$

$$\gamma = -\frac{A^3}{27}$$

$$u = \frac{-\beta \pm \sqrt{\beta^2 - 4\alpha\gamma}}{2\alpha}$$

$$u = \frac{-B \pm \sqrt{B^2 + \frac{4A^3}{27}}}{2}$$

$$\therefore t = \sqrt[3]{u} = \sqrt[3]{\frac{-B \pm \sqrt{B^2 + \frac{4A^3}{27}}}{2}}$$

Substituting into (2) yields:

$$s^3 = B + t^3 = B + \left(\sqrt[3]{\frac{-B \pm \sqrt{B^2 + \frac{4A^3}{27}}}{2}}\right)^3$$

$$\therefore s = \sqrt[3]{B + \left(\sqrt[3]{\frac{-B \pm \sqrt{B^2 + \frac{4A^3}{27}}}{2}}\right)^3}$$

Now, $y = s - t$

$$\therefore y = \sqrt[3]{B + \left(\sqrt[3]{\frac{-B \pm \sqrt{B^2 + \frac{4A^3}{27}}}{2}}\right)^3} - \sqrt[3]{\frac{-B \pm \sqrt{B^2 + \frac{4A^3}{27}}}{2}}$$

Now, $x = y - \frac{b}{3a}$

$$x = \left(\sqrt[3]{B + \left(\sqrt[3]{\frac{-B \pm \sqrt{B^2 + \frac{4A^3}{27}}}{2}}\right)^3} - \sqrt[3]{\frac{-B \pm \sqrt{B^2 + \frac{4A^3}{27}}}{2}}\right) - \frac{b}{3a}$$

Where, $A = \frac{\left(c - \frac{b^2}{3a}\right)}{a}$ & $B = -\frac{\left(d + \frac{2b^3}{27a^2} - \frac{bc}{3a}\right)}{a}$

4.2 FUNDAMENTALS OF ARITHMETIC:

Rational Numbers: Every rational number can be written as $r = \frac{(r-1)+(r+1)}{2}$

Irrational Numbers: $\lim_{m\to\infty}\lim_{n\to\infty}\left(\cos^{2n}(m!\times\pi\times x)\right) = \begin{cases} 1 & \text{x is rational} \\ 0 & \text{x is irrational} \end{cases}$

4.3 ALGEBRAIC EXPANSION:

Babylonian Identity:

$$\left(\frac{1}{2}\left(x-\frac{1}{x}\right)\right)^2 + 1 = \left(\frac{1}{2}\left(x+\frac{1}{x}\right)\right)^2$$

(c1800BC)

Common Products And Factors:

$$(a+b)^2 = a^2 + 2ab + b^2$$

$$(a-b)^2 = a^2 - 2ab + b^2$$

$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$

$$(a-b)^3 = a^3 - 3a^2b + 3ab^2 - b^3$$

$$a^2 - b^2 = (a-b)(a+b)$$

$$a^3 - b^3 = (a-b)(a^2 + ab + b^2)$$

$$a^3 + b^3 = (a+b)(a^2 - ab + b^2)$$

$$(x \pm y)^2 = x^2 \pm 2xy + y^2$$

$$(x \pm y)^3 = x^3 \pm 3x^2y + 3xy^2 \pm y^3$$

$$(x \pm y)^4 = x^4 \pm 4x^3y + 6x^2y^2 \pm 4xy^3 + y^4$$

Binomial Theorem:

For any value of n, whether positive, negative, integer or non-integer, the value of the nth power of a binomial is given by:

$$(a+b)^n = a^n + na^{n-1}b + \frac{n(n-1)}{2}a^{n-2}b^2 + \ldots + b^n$$

Binomial Expansion:

For any power of n, the binomial (a + x) can be expanded

$$(a+x)^n = a^n + na^{n-1}x + \frac{n(n-1)}{2}a^{n-2}x^2 + \ldots + x^n$$

This is particularly useful when x is very much less than a so that the first few terms provide a good approximation of the value of the expression. There will always be n+1 terms and the general form is:

$$(a+x)^n = \sum_{k=0}^{n} \frac{n!}{(n-k)!k!} a^{n-k} x^k$$

Note that the factorial is given by

$$n! = 1 \cdot 2 \cdot 3 \ldots \cdot n$$
$$0! = 1$$

Difference of two squares:

$$a^2 - b^2 = (a+b)(a-b)$$

Brahmagupta–Fibonacci Identity:

$$\begin{aligned}\left(a^2+b^2\right)\left(c^2+d^2\right) &= (ac-bd)^2 + (ad+bc)^2 \qquad (1)\\ &= (ac+bd)^2 + (ad-bc)^2 . \qquad (2)\end{aligned}$$

Also,

$$\begin{aligned}\left(a^2+nb^2\right)\left(c^2+nd^2\right) &= (ac-nbd)^2 + n\,(ad+bc)^2 \qquad (3)\\ &= (ac+nbd)^2 + n\,(ad-bc)^2 , \qquad (4)\end{aligned}$$

Degen's eight-square identity:

$$\begin{aligned}&(a_1^2+a_2^2+a_3^2+a_4^2+a_5^2+a_6^2+a_7^2+a_8^2)(b_1^2+b_2^2+b_3^2+b_4^2+b_5^2+b_6^2+b_7^2+b_8^2) =\\
&(a_1b_1 - a_2b_2 - a_3b_3 - a_4b_4 - a_5b_5 - a_6b_6 - a_7b_7 - a_8b_8)^2+\\
&(a_1b_2 + a_2b_1 + a_3b_4 - a_4b_3 + a_5b_6 - a_6b_5 - a_7b_8 + a_8b_7)^2+\\
&(a_1b_3 - a_2b_4 + a_3b_1 + a_4b_2 + a_5b_7 + a_6b_8 - a_7b_5 - a_8b_6)^2+\\
&(a_1b_4 + a_2b_3 - a_3b_2 + a_4b_1 + a_5b_8 - a_6b_7 + a_7b_6 - a_8b_5)^2+\\
&(a_1b_5 - a_2b_6 - a_3b_7 - a_4b_8 + a_5b_1 + a_6b_2 + a_7b_3 + a_8b_4)^2+\\
&(a_1b_6 + a_2b_5 - a_3b_8 + a_4b_7 - a_5b_2 + a_6b_1 - a_7b_4 + a_8b_3)^2+\\
&(a_1b_7 + a_2b_8 + a_3b_5 - a_4b_6 - a_5b_3 + a_6b_4 + a_7b_1 - a_8b_2)^2+\\
&(a_1b_8 - a_2b_7 + a_3b_6 + a_4b_5 - a_5b_4 - a_6b_3 + a_7b_2 + a_8b_1)^2\end{aligned}$$

Note that:

$$\begin{aligned}&(a_1^2+a_2^2+a_3^2+a_4^2)(b_1^2+b_2^2+b_3^2+b_4^2) =\\
&(a_1b_1 - a_2b_2 - a_3b_3 - a_4b_4)^2+\\
&(a_1b_2 + a_2b_1 + a_3b_4 - a_4b_3)^2+\\
&(a_1b_3 - a_2b_4 + a_3b_1 + a_4b_2)^2+\\
&(a_1b_4 + a_2b_3 - a_3b_2 + a_4b_1)^2\end{aligned}$$

and,

$$(a_5^2 + a_6^2 + a_7^2 + a_8^2)(b_1^2 + b_2^2 + b_3^2 + b_4^2) =$$
$$(a_5b_1 + a_6b_2 + a_7b_3 + a_8b_4)^2 +$$
$$(a_5b_2 - a_6b_1 + a_7b_4 - a_8b_3)^2 +$$
$$(a_5b_3 - a_6b_4 - a_7b_1 + a_8b_2)^2 +$$
$$(a_5b_4 + a_6b_3 - a_7b_2 - a_8b_1)^2$$

4.4 ROOT EXPANSIONS:

$$\frac{1}{k}\left(\sqrt{kx} \pm \sqrt{ky}\right)^2 = \left(\sqrt{x} \pm \sqrt{y}\right)^2$$

$$\sqrt{x} \pm \sqrt{y} = \sqrt{\frac{1}{k}\left(\sqrt{kx} \pm \sqrt{ky}\right)^2}$$

$$\sqrt{x} \pm \sqrt{y} = \sqrt{y\left(\sqrt{\frac{x}{y}} \pm 1\right)^2}$$

$$\sqrt{x} \pm \sqrt{y} = \sqrt{\frac{1}{x}\left(x \pm \sqrt{xy}\right)^2}$$

$$\sqrt{x} \pm \sqrt{y} = \sqrt{k\left(\sqrt{\frac{x}{k}} \pm \sqrt{\frac{y}{k}}\right)^2}$$

4.5 LIMIT MANIPULATIONS:

$$\lim_{n\to\infty}(a_n \pm b_n) = \left(\lim_{n\to\infty}(a_n)\right) \pm \left(\lim_{n\to\infty}(b_n)\right)$$
$$\lim_{n\to\infty}(ka_n) = k\left(\lim_{n\to\infty}(a_n)\right)$$
$$\lim_{n\to\infty}(a_nb_n) = \left(\lim_{n\to\infty}(a_n)\right)\left(\lim_{n\to\infty}(b_n)\right)$$
$$\lim_{n\to\infty}(f(a_n)) = f\left(\lim_{n\to\infty}(a_n)\right)$$

L'Hopital's Rule:

If $\lim_{x\to a}(f(x)) = \lim_{x\to a}(g(x)) = 0 \text{ or } \pm\infty$, and $\lim_{x\to a}\left(\frac{f'(x)}{g'(x)}\right)$ exists (ie $g'(x) \neq 0, x = a$), then it follows that

$$\lim_{x\to a}\left(\frac{f(x)}{g(x)}\right) = \lim_{x\to a}\left(\frac{f'(x)}{g'(x)}\right)$$

Proof:

$$\lim_{x\to a}\left(\frac{f(x)}{g(x)}\right) = \lim_{x\to a}\left(\frac{f(x) \div (x-a)}{g(x) \div (x-a)}\right) = \lim_{x\to a}\left(\frac{(f(x) - f(a)) \div (x-a)}{(g(x) - g(a)) \div (x-a)}\right)$$

$$= \frac{\lim_{x\to a}\left(\frac{(f(x) - f(a))}{(x-a)}\right)}{\lim_{x\to a}\left(\frac{(g(x) - g(a))}{(x-a)}\right)} = \lim_{x\to a}\left(\frac{f'(x)}{g'(x)}\right) = \frac{f'(a)}{g'(a)}$$

4.6 SUMMATION MANIPULATIONS:

$$\sum_{n=s}^{t} C \cdot f(n) = C \cdot \sum_{n=s}^{t} f(n)$$, where C is a constant

$$\sum_{n=s}^{t} f(n) + \sum_{n=s}^{t} g(n) = \sum_{n=s}^{t} [f(n) + g(n)]$$

$$\sum_{n=s}^{t} f(n) - \sum_{n=s}^{t} g(n) = \sum_{n=s}^{t} [f(n) - g(n)]$$

$$\sum_{n=s}^{t} f(n) = \sum_{n=s+p}^{t+p} f(n-p)$$

$$\sum_{n=s}^{j} f(n) + \sum_{n=j+1}^{t} f(n) = \sum_{n=s}^{t} f(n)$$

$$\left(\sum_{i=k_0}^{k_1} a_i\right)\left(\sum_{j=l_0}^{l_1} b_j\right) = \sum_{i=k_0}^{k_1}\sum_{j=l_0}^{l_1} a_i b_j$$

$$\sum_{i=k_0}^{k_1}\sum_{j=l_0}^{l_1} a_{i,j} = \sum_{j=l_0}^{l_1}\sum_{i=k_0}^{k_1} a_{i,j}$$

$$\sum_{n=0}^{t} f(2n) + \sum_{n=0}^{t} f(2n+1) = \sum_{n=0}^{2t+1} f(n)$$

$$\sum_{n=0}^{t}\sum_{i=0}^{z-1} f(z \cdot n + i) = \sum_{n=0}^{z \cdot t + z - 1} f(n)$$

$$\sum_{n=s}^{t} \ln f(n) = \ln \prod_{n=s}^{t} f(n)$$

$$c^{\left[\sum_{n=s}^{t} f(n)\right]} = \prod_{n=s}^{t} c^{f(n)}$$

4.7 COMMON FUNCTIONS:

Constant Function:

$y=a$ or $f(x)=a$

Graph is a horizontal line passing through the point $(0,a)$

$x=a$

Graph is a vertical line passing through the point $(a,0)$

Line/Linear Function:

$y = mx + c$

Graph is a line with point $(0,c)$ and slope m.

Where the gradient is between any two points $(x_1, y_1) \& (x_2, y_2)$

$$m = \frac{rise}{run} = \frac{y_2 - y_1}{x_2 - x_1}$$

Also, $y = y_1 + m(x - x_1)$
The equation of the line with gradient m .and passing through the point (x_1, y_1) .

Parabola/Quadratic Function:

$y = a(x-h)^2 + k$
The graph is a parabola that opens up if $a > 0$ or down if $a < 0$ and has a vertex at (h,k).

$y = ax^2 + bx + c$
The graph is a parabola that opens up if $a > 0$ or down if $a < 0$ and has a vertex at $\left(\frac{-b}{2a}, f\left(\frac{-b}{2a}\right)\right)$.

$x = ay^2 + by + c$
The graph is a parabola that opens right if $a > 0$ or left if $a < 0$ and has a vertex at $\left(g\left(\frac{-b}{2a}\right), \left(\frac{-b}{2a}\right)\right)$. This is not a function.

Circle:

$(x-h)^2 + (y-k)^2 = r^2$
Graph is a circle with radius r and center (h,k).

Ellipse:

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$

Graph is an ellipse with center (h,k) with vertices a units right/left from the center and vertices b units up/down from the center.

Hyperbola:

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$$

Graph is a hyperbola that opens left and right, has a center at (h,k) , vertices a units left/right of center and asymptotes that pass through center with slope $\pm\frac{b}{a}$.

$$\frac{(y-k)^2}{b^2} - \frac{(x-h)^2}{a^2} = 1$$

Graph is a hyperbola that opens up and down, has a center at (h,k) , vertices b units up/down from the center and asymptotes that pass through center with slope $\pm\frac{b}{a}$.

4.8 LINEAR ALGEBRA:

Vector Space Axioms:

A real vector space is a set X with a special element 0, and three operations:

- Addition: Given two elements x, y in X, one can form the sum x+y, which is also an element of X.
- Inverse: Given an element x in X, one can form the inverse -x, which is also an element of X.
- Scalar multiplication: Given an element x in X and a real number c, one can form the product cx, which is also an element of X.

These operations must satisfy the following axioms:

Additive axioms. For every x,y,z in X, we have

1. x+y = y+x.
2. (x+y)+z = x+(y+z).
3. 0+x = x+0 = x.
4. (-x) + x = x + (-x) = 0.

Multiplicative axioms. For every x in X and real numbers c,d, we have

5. 0x = 0
6. 1x = x
7. (cd)x = c(dx)

Distributive axioms. For every x,y in X and real numbers c,d, we have

8. c(x+y) = cx + cy.
9. (c+d)x = cx +dx.

A normed real vector space is a real vector space X with an additional operation:

- Norm: Given an element x in X, one can form the norm ||x||, which is a non-negative number.

This norm must satisfy the following axioms, for any x,y in X and any real number c:

10. ||x|| = 0 if and only if x = 0.
11. || cx || = |c| ||x||.
12. || x+y || <= ||x|| + ||y||

A complex vector space consists of the same set of axioms as the real case, but elements within the vector space are complex. The axioms are adjusted to suit.

Subspace: When the subspace is a subset of another vector space, only axioms (a) and (b) need to be proved to show that the subspace is also a vector space.

Common Spaces:

Real Numbers: $\Re, \Re^2, \Re^3, ..., \Re^n$ (n denotes dimension)

Complex Numbers: $C, C^2, C^3, ..., C^n$ (n denotes dimension)

Polynomials: $P_1, P_2, P_3, ..., P_n$ (n denotes the highest order of x)

All continuous functions: $C[a,b]$(a & b denote the interval) (This is never a vector space as it has infinite dimensions)

Rowspace of a spanning set in R^n

Stack vectors in a matrix in rows
Use elementary row operations to put matrix into row echelon form
The non zero rows form a basis of the vector space

Columnspace of a spanning set in R^n

Stack vectors in a matrix in columns
Use elementary row operations to put matrix into row echelon form
Columns with leading entries correspond to the subset of vectors in the set that form a basis

Nullspace:

Solutions to $A\underline{x} = \underline{0}$ A

Using elementary row operations to put matrix into row echelon form, columns with no leading entries are assigned a constant and the remaining variables are solved with respect to these constants.

Nullity:

The dimension of the nullspace

$Columns(A) = Nullity(A) + Rank(A)$

Linear Dependence:

$c_1 r_1 + c_2 r_2 + ... + c_n r_n = 0$

Then, $c_1 = c_2 = c_n = 0$

If the trivial solution is the only solution, $r_1, r_2, ... r_n$ are independent.

$r(A) \neq r(A|b)$: No Solution

$r(A) = r(A|b) = n$: Unique Solution

$r(A) = r(A|b) < n$: Infinite Solutions

Basis: S is a basis of V if:

S spans V

S is linearly dependant

$S = \{u_1, u_2, u_3, ..., u_n\}$

The general vector within the vector space is: $w = \begin{bmatrix} x \\ y \\ z \\ ... \end{bmatrix}$

$$w = c_1u_1 + c_2u_2 + c_3u_3 + ... + c_nu_n$$

Therefore, $[w] = \begin{bmatrix} u_{11} & u_{21} & u_{31} & ... & u_{n1} \\ u_{12} & u_{22} & u_{32} & ... & u_{n2} \\ u_{13} & u_{23} & u_{33} & ... & u_{n3} \\ ... & ... & ... & ... & ... \\ u_{1n} & u_{2n} & u_{3n} & ... & u_{nn} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ ... \\ c_n \end{bmatrix}$

If the determinant of the square matrix is not zero, the matrix is invertible. Therefore, the solution is unique. Hence, all vectors in w are linear combinations of S. Because of this, S spans w.

Standard Basis:

Real Numbers $S(\Re^n) = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ ... \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ ... \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ ... \\ 0 \end{bmatrix}, ..., \begin{bmatrix} 0 \\ 0 \\ 0 \\ ... \\ 1 \end{bmatrix} \right\}$

Polynomials $S(P_n) = \{1, x, x^2, x^3, ..., x^n\}$

Any set the forms the basis of a vector space must contain the same number of linearly independent vectors as the standard basis.

Orthogonal Complement:

$W^{\perp}$ is the nullspace of A, where A is the matrix that contains $\{v_1, v_2, v_3, ..., v_n\}$ in rows.

$$\dim(W^{\perp}) = nullity(A)$$

Orthonormal Basis:

A basis of mutually orthogonal vectors of length 1. Basis can be found with the Gram-Schmidt process outline below.

$$< v_i, v_j >= \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$$

In an orthonormal basis: $u =< u, v_1 > v_1 + < u, v_2 > v_2 + < u, v_3 > v_3 + ... + < u, v_n > v_n)$
$u = c_1v_1 + c_2v_2 + c_3v_3 + ... + c_nv_n)$

Gram-Schmidt Process:

This finds an orthonormal basis recursively.

In a basis $B = \{u_1, u_2, u_3, ..., u_n\}$

$$q_1 = u_1$$

$$v_1 = \hat{q}_1 = \frac{q_1}{\|q_1\|}$$

Next vector needs to be orthogonal to v_1,

$$q_2 = u_2 - < u_2, v_1 > v_1$$

Similarly

$$q_3 = u_3 - < u_3, v_1 > v_1 - < u_3, v_2 > v_2$$

$$q_n = u_n - < u_3, v_1 > v_1 - < u_3, v_2 > v_2 - ... - < u_3, v_n > v_n$$

$$v_n = \hat{q}_n = \frac{q_n}{\|q_n\|}$$

Therefore, orthogonal basis is: $B' = \{v_1, v_2, v_3, ..., v_n\}$

Coordinate Vector:

If $\underline{v} = c_1 e_1 + c_2 e_2 + ... + c_n e_n$

$$\underline{v}_B = \begin{bmatrix} c_1 \\ c_2 \\ ... \\ c_n \end{bmatrix}$$

For a fixed basis (usually the standard basis) there is 1 to 1 correspondence between vectors and coordinate vectors.
Hence, a basis can be found in R^n and then translated back into the general vector space.

Dimension:

Real Numbers	$\dim(\Re^n) = n$
Polynomials	$\dim(P_n) = n + 1$
Matricis	$\dim(M_{p,q}) = p \times q$

If you know the dimensions and you are checking if a set forms a basis of the vector space, only Linear Independence or Span needs to be checked.

4.9 COMPLEX VECTOR SPACES:

Form: $C^n = \begin{bmatrix} a_1 + ib_1 \\ a_2 + ib_2 \\ ... \\ a_n + ib_n \end{bmatrix}$

Dot Product:

$$u \bullet v = \bar{u}_1 v_1 + \bar{u}_2 v_2 + ... + \bar{u}_n v_n$$

Where:

$$u \bullet v = \overline{v \bullet u} \neq v \bullet u$$
$$(u+v) \bullet w = u \bullet w + v \bullet w$$
$$su \bullet v = \bar{s}(u \bullet v), s \in C$$
$$u \bullet u \geq 0$$
$$u \bullet u = 0 \text{ iff } u = 0$$

Inner Product:

$$\|u\| = \sqrt{u \bullet u} = \sqrt{|u_1|^2 + |u_2|^2 + \ldots + |u_n|^2}$$
$$d(u,v) = \|u - v\|$$

Orthogonal if $u \bullet v = 0$
Parallel if $u = sv, s \in C$

4.10 LINEAR TRANSITIONS & TRANSFORMATIONS:

Transition Matrix: From 1 vector space to another vector space

$$T(u) = T(c_1u_1 + c_2u_2 + c_3u_3 + \ldots + c_nu_n)$$
$$T(u) = c_1T(u_1) + c_2T(u_2) + c_3T(u_3) + \ldots + c_nT(u_n)$$

Nullity(T)+Rank(T)=Dim(V)=Columns(T)

Change of Basis Transition Matrix:

$$v_{B'} = M_{B'}{}^{-1} M_B v_B$$
$$v_{B'} = C_{BB'} v_B$$

For a general vector space with the standard basis: $S = \{s_1, s_2, \ldots, s_n\}$

$$M_B = [(v_1)_S \,|\, \ldots \,|\, (v_n)_S]$$
$$M_{B'} = [(u_1)_S \,|\, \ldots \,|\, (u_m)_S]$$

Transformation Matrix: From 1 basis to another basis

$$V = span(\{v_1, v_2, v_3, \ldots, v_n\})$$
$$B_1 = \{v_1, v_2, v_3, \ldots, v_n\}$$

$$U = span(\{u_1, u_2, u_3, \ldots, u_m\})$$
$$B_2 = \{u_1, u_2, u_3, \ldots, u_m\}$$

$$A = [(T(v_1))_{B2} \,|\, (T(v_2))_{B2} \,|\, \ldots \,|\, (T(v_n))_{B2}]$$
$$A' = C_{B'B}{}^{-1} A C_{B'B}$$

4.11 INNER PRODUCTS:

Definition: An extension of the dot product into a general vector space.

Axioms:

1. $<u,v>=<v,u>$
2. $<u,v+w>=<u,v>+<u,w>$
3. $<ku,v>=k<u,v>$

4. $< u,u > \geq 0$

$< u,u > = 0 \text{ iff } u = 0$

Unit Vector: $\hat{u} = \frac{u}{\|u\|}$

Cavchy-Schuarz Inequality: $< u,v >^2 \leq < u,u > \times < v,v >$

Inner Product Space:

$$\|u\| = < u,u >^{\frac{1}{2}} = \sqrt{< u,u >}$$

$$\|u\|^2 = < u,u >$$

$$< u,v >^2 \leq \|u\|^2 \times \|v\|^2 \Rightarrow \left(\frac{< u,v >}{\|u\|\|v\|} \right)^2 \leq 1 \Rightarrow -1 \leq \frac{< u,v >}{\|u\|\|v\|} \leq 1$$

$$\|u\| \geq 0, \|u\| = 0 \text{ iff } u = 0$$

$$\|ku\| = |k|\|u\|$$

$$\|u+v\| = \|u\| + \|v\|$$

Angle between two vectors:

As defined by the inner product,

$$\cos(\theta) = \frac{< u,v >}{\|u\|\|v\|}$$

Orthogonal if: $< u,v > = 0$

Distance between two vectors:

As defined by the inner product,

$$d(u,v) = \|u-v\|$$

Generalised Pythagoras for orthogonal vectors:

$$\|u+v\|^2 = \|u\|^2 + \|v\|^2$$

4.12 PRIME NUMBERS:

Determinate: $$\Delta(N) = \left\lfloor \frac{1 + \left\lfloor \frac{3}{N} \right\rfloor}{1 + \sum_{k=1}^{\left\lfloor \frac{\sqrt{N}+1}{2} \right\rfloor} \left\lfloor \frac{2k+1}{N} \times \left\lfloor \frac{N}{2k+1} \right\rfloor \right\rfloor} \right\rfloor = \begin{cases} 1 & \text{if N is odd and prime} \\ 0 & \text{if N is odd and composite} \end{cases}$$

List of Prime Numbers:

2	3	5	7	11	13	17	19	23	29	31	37	41	43	47	53	59	61	67	71
73	79	83	89	97	101	103	107	109	113	127	131	137	139	149	151	157	163	167	173
179	181	191	193	197	199	211	223	227	229	233	239	241	251	257	263	269	271	277	281
283	293	307	311	313	317	331	337	347	349	353	359	367	373	379	383	389	397	401	409
419	421	431	433	439	443	449	457	461	463	467	479	487	491	499	503	509	521	523	541
547	557	563	569	571	577	587	593	599	601	607	613	617	619	631	641	643	647	653	659
661	673	677	683	691	701	709	719	727	733	739	743	751	757	761	769	773	787	797	809
811	821	823	827	829	839	853	857	859	863	877	881	883	887	907	911	919	929	937	941

947	953	967	971	977	983	991	997	1009	1013	1019	1021	1031	1033	1039	1049	1051	1061	1063	1069
1087	1091	1093	1097	1103	1109	1117	1123	1129	1151	1153	1163	1171	1181	1187	1193	1201	1213	1217	1223
1229	1231	1237	1249	1259	1277	1279	1283	1289	1291	1297	1301	1303	1307	1319	1321	1327	1361	1367	1373
1381	1399	1409	1423	1427	1429	1433	1439	1447	1451	1453	1459	1471	1481	1483	1487	1489	1493	1499	1511
1523	1531	1543	1549	1553	1559	1567	1571	1579	1583	1597	1601	1607	1609	1613	1619	1621	1627	1637	1657
1663	1667	1669	1693	1697	1699	1709	1721	1723	1733	1741	1747	1753	1759	1777	1783	1787	1789	1801	1811
1823	1831	1847	1861	1867	1871	1873	1877	1879	1889	1901	1907	1913	1931	1933	1949	1951	1973	1979	1987
1993	1997	1999	2003	2011	2017	2027	2029	2039	2053	2063	2069	2081	2083	2087	2089	2099	2111	2113	2129
2131	2137	2141	2143	2153	2161	2179	2203	2207	2213	2221	2237	2239	2243	2251	2267	2269	2273	2281	2287
2293	2297	2309	2311	2333	2339	2341	2347	2351	2357	2371	2377	2381	2383	2389	2393	2399	2411	2417	2423
2437	2441	2447	2459	2467	2473	2477	2503	2521	2531	2539	2543	2549	2551	2557	2579	2591	2593	2609	2617
2621	2633	2647	2657	2659	2663	2671	2677	2683	2687	2689	2693	2699	2707	2711	2713	2719	2729	2731	2741
2749	2753	2767	2777	2789	2791	2797	2801	2803	2819	2833	2837	2843	2851	2857	2861	2879	2887	2897	2903
2909	2917	2927	2939	2953	2957	2963	2969	2971	2999	3001	3011	3019	3023	3037	3041	3049	3061	3067	3079
3083	3089	3109	3119	3121	3137	3163	3167	3169	3181	3187	3191	3203	3209	3217	3221	3229	3251	3253	3257
3259	3271	3299	3301	3307	3313	3319	3323	3329	3331	3343	3347	3359	3361	3371	3373	3389	3391	3407	3413
3433	3449	3457	3461	3463	3467	3469	3491	3499	3511	3517	3527	3529	3533	3539	3541	3547	3557	3559	3571

Fundamental Theory of Arithmetic:

That every integer greater than 1 is either prime itself or is the product of a finite number of prime numbers.

Laprange's Theorem:

That every natural number can be written as the sum of four square integers.

Eg: $59 = 7^2 + 3^2 + 1^2 + 0^2$

Ie: $x = a^2 + b^2 + c^2 + d^2; a, b, c, d, x \in N_0$

Additive primes:

Primes: such that the sum of digits is a prime.

2, 3, 5, 7, 11, 23, 29, 41, 43, 47, 61, 67, 83, 89, 101, 113, 131…

Annihilating primes:

Primes: such that d(p) = 0, where d(p) is the shadow of a sequence of natural numbers

3, 7, 11, 17, 47, 53, 61, 67, 73, 79, 89, 101, 139, 151, 157, 191, 199

Bell number primes:

Primes: that are the number of partitions of a set with n members.

2, 5, 877, 27644437, 35742549198872617291353508656626642567, 359334085968622831041960188598043661065388726959079837. The next term has 6,539 digits.

Carol primes:

Of the form $(2^n-1)^2 - 2$.

7, 47, 223, 3967, 16127, 1046527, 16769023, 1073676287, 68718952447, 274876858367, 4398042316799, 1125899839733759, 18014398241046527, 1298074214633706835075030044377087

Centered decagonal primes:

Of the form $5(n^2 - n) + 1$.

11, 31, 61, 101, 151, 211, 281, 661, 911, 1051, 1201, 1361, 1531, 1901, 2311, 2531, 3001, 3251, 3511, 4651, 5281, 6301, 6661, 7411, 9461, 9901, 12251, 13781, 14851, 15401, 18301, 18911, 19531, 20161, 22111, 24151, 24851, 25561, 27011, 27751

Centered heptagonal primes:

Of the form $(7n^2 - 7n + 2) / 2$.

43, 71, 197, 463, 547, 953, 1471, 1933, 2647, 2843, 3697, 4663, 5741, 8233, 9283, 10781, 11173, 12391, 14561, 18397, 20483, 29303, 29947, 34651, 37493, 41203, 46691, 50821, 54251, 56897, 57793, 65213, 68111, 72073, 76147, 84631, 89041, 93563

Centered square primes:

Of the form $n^2 + (n+1)^2$.
5, 13, 41, 61, 113, 181, 313, 421, 613, 761, 1013, 1201, 1301, 1741, 1861, 2113, 2381, 2521, 3121, 3613, 4513, 5101, 7321, 8581, 9661, 9941, 10513, 12641, 13613, 14281, 14621, 15313, 16381, 19013, 19801, 20201, 21013, 21841, 23981, 24421, 26681

Centered triangular primes:

Of the form $(3n^2 + 3n + 2) / 2$.
19, 31, 109, 199, 409, 571, 631, 829, 1489, 1999, 2341, 2971, 3529, 4621, 4789, 7039, 7669, 8779, 9721, 10459, 10711, 13681, 14851, 16069, 16381, 17659, 20011, 20359, 23251, 25939, 27541, 29191, 29611, 31321, 34429, 36739, 40099, 40591, 42589

Chen primes:

Where p is prime and p+2 is either a prime or semiprime.
2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 47, 53, 59, 67, 71, 83, 89, 101, 107, 109, 113, 127, 131, 137, 139, 149, 157, 167, 179, 181, 191, 197, 199, 211, 227, 233, 239, 251, 257, 263, 269, 281, 293, 307, 311, 317, 337, 347, 353, 359, 379, 389, 401, 409

Circular primes:

A circular prime number is a number that remains prime on any cyclic rotation of its digits (in base 10).
2, 3, 5, 7, 11, 13, 17, 31, 37, 71, 73, 79, 97, 113, 131, 197, 199, 311, 337, 373, 719, 733, 919, 971, 991, 1193, 1931, 3119, 3779, 7793, 7937, 9311, 9377, 11939, 19391, 19937, 37199, 39119, 71993, 91193, 93719, 93911, 99371, 193939, 199933, 319993, 331999, 391939, 393919, 919393, 933199, 939193, 939391, 993319, 999331

Cousin primes:

Where (p, p+4) are both prime.
(3, 7), (7, 11), (13, 17), (19, 23), (37, 41), (43, 47), (67, 71), (79, 83), (97, 101), (103, 107), (109, 113), (127, 131), (163, 167), (193, 197), (223, 227), (229, 233), (277, 281)

Cuban primes:

Of the form $\frac{x^3 - y^3}{x - y}, x = y + 1 \Rightarrow (y+1)^3 - y^3$
7, 19, 37, 61, 127, 271, 331, 397, 547, 631, 919, 1657, 1801, 1951, 2269, 2437, 2791, 3169, 3571, 4219, 4447, 5167, 5419, 6211, 7057, 7351, 8269, 9241, 10267, 11719, 12097, 13267, 13669, 16651, 19441, 19927, 22447, 23497, 24571, 25117, 26227, 27361, 33391, 35317

Of the form $\frac{x^3 - y^3}{x - y}, x = y + 2 \Rightarrow \frac{(y+2)^3 - y^3}{2}$
13, 109, 193, 433, 769, 1201, 1453, 2029, 3469, 3889, 4801, 10093, 12289, 13873, 18253, 20173, 21169, 22189, 28813, 37633, 43201, 47629, 60493, 63949, 65713, 69313, 73009, 76801, 84673, 106033, 108301, 112909, 115249

Cullen primes:

Of the form $n \times 2^n + 1$.
3, 393050634124102232869567034555427371542904833

Dihedral primes:

Primes: that remain prime when read upside down or mirrored in a seven-segment display.
2, 5, 11, 101, 181, 1181, 1811, 18181, 108881, 110881, 118081, 120121, 121021, 121151, 150151, 151051, 151121, 180181, 180811, 181081

Double factorial primes:

Of the form n!! + 1. Values of n:

0, 1, 2, 518, 33416, 37310, 52608

Of the form n!! − 1. Values of n:
3, 4, 6, 8, 16, 26, 64, 82, 90, 118, 194, 214, 728, 842, 888, 2328, 3326, 6404, 8670, 9682, 27056, 44318

Double Mersenne primes:

A subset of Mersenne primes: of the form $2^{2^{p-1}-1}-1$ for prime *p*.
7, 127, 2147483647, 170141183460469231731687303715884105727

Eisenstein primes without imaginary part:

Eisenstein integers that are irreducible and real numbers (primes: of the form 3n − 1).
2, 5, 11, 17, 23, 29, 41, 47, 53, 59, 71, 83, 89, 101, 107, 113, 131, 137, 149, 167, 173, 179, 191, 197, 227, 233, 239, 251, 257, 263, 269, 281, 293, 311, 317, 347, 353, 359, 383, 389, 401

Emirps:

Primes which become a different prime when their decimal digits are reversed.
13, 17, 31, 37, 71, 73, 79, 97, 107, 113, 149, 157, 167, 179, 199, 311, 337, 347, 359, 389, 701, 709, 733, 739, 743, 751, 761, 769, 907, 937, 941, 953, 967, 971, 983, 991

Euclid primes:

Of the form $p_n\# + 1$ (a subset of primorial prime:).
3, 7, 31, 211, 2311, 200560490131

Even prime:

Of the form 2n.
2

Factorial primes:

Of the form *n*! − 1 or *n*! + 1.
2, 3, 5, 7, 23, 719, 5039, 39916801, 479001599, 87178291199, 10888869450418352160768000001, 265252859812191058636308479999999, 263130836933693530167218012159999999, 8683317618811886495518194401279999999

Fermat primes:

Of the form $2^{2^{n+1}}+1$
3, 5, 17, 257, 65537

Fibonacci primes:

Primes in the Fibonacci sequence
2, 3, 5, 13, 89, 233, 1597, 28657, 514229, 433494437, 2971215073, 99194853094755497, 1066340417491710595814572169, 19134702400093278081449423917

Fortunate primes:

Fortunate numbers that are prime
3, 5, 7, 13, 17, 19, 23, 37, 47, 59, 61, 67, 71, 79, 89, 101, 103, 107, 109, 127, 151, 157, 163, 167, 191, 197, 199, 223, 229, 233, 239, 271, 277, 283, 293, 307, 311, 313, 331, 353, 373, 379, 383, 397

Gaussian primes:

Prime elements of the Gaussian integers (primes: of the form 4n + 3).

3, 7, 11, 19, 23, 31, 43, 47, 59, 67, 71, 79, 83, 103, 107, 127, 131, 139, 151, 163, 167, 179, 191, 199, 211, 223, 227, 239, 251, 263, 271, 283, 307, 311, 331, 347, 359, 367, 379, 383, 419, 431, 439, 443, 463, 467, 479, 487, 491, 499, 503

Generalized Fermat primes base 10:

Of the form $10^n + 1$, where $n > 0$.
11, 101

Genocchi number primes:
The only positive prime Genocchi number is 17

Gilda's primes:
Gilda's numbers that are prime.
29, 683, 997, 2207, 30571351

Good primes:
Primes: p_n for which $p_n^2 > p_{n-i}\, p_{n+i}$ for all $1 \le i \le n-1$, where p_n is the nth prime.
5, 11, 17, 29, 37, 41, 53, 59, 67, 71, 97, 101, 127, 149, 179, 191, 223, 227, 251, 257, 269, 307

Happy primes:
Happy numbers that are prime.
7, 13, 19, 23, 31, 79, 97, 103, 109, 139, 167, 193, 239, 263, 293, 313, 331, 367, 379, 383, 397, 409, 487, 563, 617, 653, 673, 683, 709, 739, 761, 863, 881, 907, 937, 1009, 1033, 1039, 1093

Harmonic primes:
Primes p for which there are no solutions to $H_k \equiv 0 \pmod{p}$ and $H_k \equiv -\omega_p \pmod{p}$ for $1 \le k \le p-2$, where $\omega_p$is the Wolstenholme quotient.
5, 13, 17, 23, 41, 67, 73, 79, 107, 113, 139, 149, 157, 179, 191, 193, 223, 239, 241, 251, 263, 277, 281, 293, 307, 311, 317, 331, 337, 349

Higgs primes for squares:
Primes p for which p−1 divides the square of the product of all earlier terms.
2, 3, 5, 7, 11, 13, 19, 23, 29, 31, 37, 43, 47, 53, 59, 61, 67, 71, 79, 101, 107, 127, 131, 139, 149, 151, 157, 173, 181, 191, 197, 199, 211, 223, 229, 263, 269, 277, 283, 311, 317, 331, 347, 349

Highly cototient number primes:
Primes that are a cototient more often than any integer below it except 1.
2, 23, 47, 59, 83, 89, 113, 167, 269, 389, 419, 509, 659, 839, 1049, 1259, 1889

Irregular primes:
Odd primes p which divide the class number of the p-th cyclotomic field.
37, 59, 67, 101, 103, 131, 149, 157, 233, 257, 263, 271, 283, 293, 307, 311, 347, 353, 379, 389, 401, 409, 421, 433, 461, 463, 467, 491, 523, 541, 547, 557, 577, 587, 593, 607, 613, 617, 619

(p, p−5) irregular primes:
Primes p such that (p, p−5) is an irregular pair.
37

(p, p−9) irregular primes:
Primes p such that (p, p−9) is an irregular pair
67, 877

Isolated primes:
Primes p such that neither p−2 nor p+2 is prime.
2, 23, 37, 47, 53, 67, 79, 83, 89, 97, 113, 127, 131, 157, 163, 167, 173, 211, 223, 233, 251, 257, 263, 277, 293, 307, 317, 331, 337, 353, 359, 367, 373, 379, 383, 389, 397, 401, 409, 439, 443, 449, 457, 467, 479, 487, 491, 499, 503, 509, 541, 547, 557, 563, 577, 587, 593, 607, 613, 631, 647, 653, 673, 677, 683, 691, 701, 709, 719, 727, 733, 739, 743, 751, 757, 761, 769, 773, 787, 797, 839, 853, 863, 877, 887, 907, 911, 919, 929, 937, 941, 947, 953, 967, 971, 977, 983, 991, 997

Kynea primes:
Of the form $(2^n + 1)^2 - 2$.
7, 23, 79, 1087, 66047, 263167, 16785407, 1073807359, 17180131327, 68720001023, 4398050705407, 70368760954879, 18014398777917439, 18446744082299486207

Left-truncatable primes:
Primes that remain prime when the leading decimal digit is successively removed.
2, 3, 5, 7, 13, 17, 23, 37, 43, 47, 53, 67, 73, 83, 97, 113, 137, 167, 173, 197, 223, 283, 313, 317, 337, 347, 353, 367, 373, 383, 397, 443, 467, 523, 547, 613, 617, 643, 647, 653, 673, 683

Leyland primes:
Of the form $x^y + y^x$, with $1 < x \leq y$.
17, 593, 32993, 2097593, 8589935681, 59604644783353249, 523347633027360537213687137, 43143988327398957279342419750374600193

Long primes:
Primes p for which, in a given base b, $\frac{b^{p-1}-1}{p}$ gives a cyclic number. They are also called full reptend primes:.
Primes p for base 10:
7, 17, 19, 23, 29, 47, 59, 61, 97, 109, 113, 131, 149, 167, 179, 181, 193, 223, 229, 233, 257, 263, 269, 313, 337, 367, 379, 383, 389, 419, 433, 461, 487, 491, 499, 503, 509, 541, 571, 577, 593

Lucas primes:
Primes in the Lucas number sequence $L_0 = 2$, $L_1 = 1$, $L_n = L_{n-1} + L_{n-2}$.
2, 3, 7, 11, 29, 47, 199, 521, 2207, 3571, 9349, 3010349, 54018521, 370248451, 6643838879, 119218851371, 5600748293801, 688846502588399, 32361122672259149

Lucky primes:
Lucky numbers that are prime.
3, 7, 13, 31, 37, 43, 67, 73, 79, 127, 151, 163, 193, 211, 223, 241, 283, 307, 331, 349, 367, 409, 421, 433, 463, 487, 541, 577, 601, 613, 619, 631, 643, 673, 727, 739, 769, 787, 823, 883, 937, 991, 997

Markov primes:
Primes p for which there exist integers x and y such that $x^2 + y^2 + p^2 = 3xyp$.
2, 5, 13, 29, 89, 233, 433, 1597, 2897, 5741, 7561, 28657, 33461, 43261, 96557, 426389, 514229, 1686049, 2922509, 3276509, 94418953, 321534781, 433494437, 780291637, 1405695061, 2971215073, 19577194573, 25209506681

Mersenne primes:
Of the form $2^n - 1$.
3, 7, 31, 127, 8191, 131071, 524287, 2147483647, 2305843009213693951, 618970019642690137449562111, 162259276829213363391578010288127, 170141183460469231731687303715884105727

Mersenne prime exponents:
Primes p such that $2^p - 1$ is prime.
2, 3, 5, 7, 13, 17, 19, 31, 61, 89, 107, 127, 521, 607, 1279, 2203, 2281, 3217, 4253, 4423, 9689, 9941, 11213, 19937, 21701, 23209, 44497, 86243, 110503, 132049, 216091, 756839, 859433, 1257787, 1398269, 2976221, 3021377, 6972593, 13466917, 20996011, 24036583

Mills primes:
Of the form $\lfloor \theta^{3^n} \rfloor$, where θ is Mills' constant. This form is prime for all positive integers n.
2, 11, 1361, 2521008887, 1602223620400981813183132018

Minimal primes:
Primes for which there is no shorter sub-sequence of the decimal digits that form a prime. There are exactly 26 minimal primes::
2, 3, 5, 7, 11, 19, 41, 61, 89, 409, 449, 499, 881, 991, 6469, 6949, 9001, 9049, 9649, 9949, 60649, 666649, 946669, 60000049, 66000049, 66600049

Motzkin primes:
Primes that are the number of different ways of drawing non-intersecting chords on a circle between n points.
2, 127, 15511, 953467954114363

Newman–Shanks–Williams primes:
Newman–Shanks–Williams numbers that are prime.
7, 41, 239, 9369319, 63018038201, 489133282872437279, 19175002942688032928599

Non-generous primes:
Primes p for which the least positive primitive root is not a primitive root of p^2.
2, 40487, 6692367337

Odd primes:
Of the form 2n − 1.
3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89, 97, 101, 103, 107, 109, 113, 127, 131, 137, 139, 149, 151, 157, 163, 167, 173, 179, 181, 191, 193, 197, 199...

Padovan primes:
Primes in the Padovan sequence $P(0) = P(1) = P(2) = 1$, $P(n) = P(n-2) + P(n-3)$.
2, 3, 5, 7, 37, 151, 3329, 23833, 13091204281, 3093215881333057, 1363005552434666078217421284621279933627102780881053358473

Palindromic primes:
Primes that remain the same when their decimal digits are read backwards.
2, 3, 5, 7, 11, 101, 131, 151, 181, 191, 313, 353, 373, 383, 727, 757, 787, 797, 919, 929, 10301, 10501, 10601, 11311, 11411, 12421, 12721, 12821, 13331, 13831, 13931, 14341, 14741

Palindromic wing primes:
Primes of the form $\frac{a(10^m - 1)}{9} \pm b \times 10^{\frac{m}{2}}$
101, 131, 151, 181, 191, 313, 353, 373, 383, 727, 757, 787, 797, 919, 929, 11311, 11411, 33533, 77377, 77477, 77977, 1114111, 1117111, 3331333, 3337333, 7772777, 7774777, 7778777, 111181111, 111191111, 777767777, 77777677777, 99999199999

Partition primes:
Partition numbers that are prime.
2, 3, 5, 7, 11, 101, 17977, 10619863, 6620830889, 80630964769, 228204732751, 1171432692373, 1398341745571, 10963707205259, 15285151248481, 10657331232548839, 790738119649411319, 18987964267331664557

Pell primes:
Primes in the Pell number sequence $P_0 = 0$, $P_1 = 1$, $P_n = 2P_{n-1} + P_{n-2}$.
2, 5, 29, 5741, 33461, 44560482149, 1746860020068409, 68480406462161287469, 13558774610046711780701, 4125636888562548868221559797461449

Permutable primes:
Any permutation of the decimal digits is a prime.
2, 3, 5, 7, 11, 13, 17, 31, 37, 71, 73, 79, 97, 113, 131, 199, 311, 337, 373, 733, 919, 991, 1111111111111111111, 11111111111111111111111

Perrin primes:
Primes in the Perrin number sequence P(0) = 3, P(1) = 0, P(2) = 2, P(n) = P(n−2) + P(n−3).
2, 3, 5, 7, 17, 29, 277, 367, 853, 14197, 43721, 1442968193, 792606555396977, 187278659180417234321, 66241160488780141071579864797

Pierpont primes:
Of the form $2^u3^v + 1$ for some integers $u,v \geq 0$.
These are also class 1- primes:.
2, 3, 5, 7, 13, 17, 19, 37, 73, 97, 109, 163, 193, 257, 433, 487, 577, 769, 1153, 1297, 1459, 2593, 2917, 3457, 3889, 10369, 12289, 17497, 18433, 39367, 52489, 65537, 139969, 147457

Pillai primes:

Primes p for which there exist $n > 0$ such that p divides $n! + 1$ and n does not divide $p-1$.
23, 29, 59, 61, 67, 71, 79, 83, 109, 137, 139, 149, 193, 227, 233, 239, 251, 257, 269, 271, 277, 293, 307, 311, 317, 359, 379, 383, 389, 397, 401, 419, 431, 449, 461, 463, 467, 479, 499

Primeval primes:

Primes for which there are more prime permutations of some or all the decimal digits than for any smaller number.
2, 13, 37, 107, 113, 137, 1013, 1237, 1367, 10079

Primorial primes:

Of the form $p_n\# -1$ or $p_n\# + 1$.
3, 5, 7, 29, 31, 211, 2309, 2311, 30029, 200560490131, 304250263527209, 23768741896345550770650537601358309

Proth primes:

Of the form $k \times 2^n + 1$, with odd k and $k < 2^n$.
3, 5, 13, 17, 41, 97, 113, 193, 241, 257, 353, 449, 577, 641, 673, 769, 929, 1153, 1217, 1409, 1601, 2113, 2689, 2753, 3137, 3329, 3457, 4481, 4993, 6529, 7297, 7681, 7937, 9473, 9601, 9857

Pythagorean primes:

Of the form 4n + 1.
5, 13, 17, 29, 37, 41, 53, 61, 73, 89, 97, 101, 109, 113, 137, 149, 157, 173, 181, 193, 197, 229, 233, 241, 257, 269, 277, 281, 293, 313, 317, 337, 349, 353, 373, 389, 397, 401, 409, 421, 433, 449

Prime quadruplets:

Where (p, p+2, p+6, p+8) are all prime.
(5, 7, 11, 13), (11, 13, 17, 19), (101, 103, 107, 109), (191, 193, 197, 199), (821, 823, 827, 829), (1481, 1483, 1487, 1489), (1871, 1873, 1877, 1879), (2081, 2083, 2087, 2089), (3251, 3253, 3257, 3259), (3461, 3463, 3467, 3469), (5651, 5653, 5657, 5659), (9431, 9433, 9437, 9439)

Primes of binary quadratic form:

Of the form $x^2 + xy + 2y^2$, with non-negative integers x and y.
2, 11, 23, 37, 43, 53, 71, 79, 107, 109, 127, 137, 149, 151, 163, 193, 197, 211, 233, 239, 263, 281, 317, 331, 337, 373, 389, 401, 421, 431, 443, 463, 487, 491, 499, 541, 547, 557, 569, 599, 613, 617, 641, 653, 659, 673, 683, 739, 743, 751, 757, 809, 821

Quartan primes:

Of the form $x^4 + y^4$, where $x,y > 0$.
2, 17, 97, 257, 337, 641, 881

Ramanujan primes:

Integers R_n that are the smallest to give at least n primes: from $x/2$ to x for all $x \geq R_n$ (all such integers are primes:).
2, 11, 17, 29, 41, 47, 59, 67, 71, 97, 101, 107, 127, 149, 151, 167, 179, 181, 227, 229, 233, 239, 241, 263, 269, 281, 307, 311, 347, 349, 367, 373, 401, 409, 419, 431, 433, 439, 461, 487, 491

Regular primes:

Primes p which do not divide the class number of the p-th cyclotomic field.
3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 41, 43, 47, 53, 61, 71, 73, 79, 83, 89, 97, 107, 109, 113, 127, 137, 139, 151, 163, 167, 173, 179, 181, 191, 193, 197, 199, 211, 223, 227, 229, 239, 241, 251, 269, 277, 281

Repunit primes:

Primes containing only the decimal digit 1.
11, 1111111111111111111, 11111111111111111111111
The next have 317 and 1,031 digits.

Primes in residue classes:

Of the form an + d for fixed a and d. Also called primes: congruent to d modulo a.

Three cases have their own entry: 2n+1 are the odd primes:, 4n+1 are Pythagorean primes:, 4n+3 are the integer Gaussian primes:.

2n+1: 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53
4n+1: 5, 13, 17, 29, 37, 41, 53, 61, 73, 89, 97, 101, 109, 113, 137
4n+3: 3, 7, 11, 19, 23, 31, 43, 47, 59, 67, 71, 79, 83, 103, 107
6n+1: 7, 13, 19, 31, 37, 43, 61, 67, 73, 79, 97, 103, 109, 127, 139
6n+5: 5, 11, 17, 23, 29, 41, 47, 53, 59, 71, 83, 89, 101, 107, 113
8n+1: 17, 41, 73, 89, 97, 113, 137, 193, 233, 241, 257, 281, 313, 337, 353
8n+3: 3, 11, 19, 43, 59, 67, 83, 107, 131, 139, 163, 179, 211, 227, 251
8n+5: 5, 13, 29, 37, 53, 61, 101, 109, 149, 157, 173, 181, 197, 229, 269
8n+7: 7, 23, 31, 47, 71, 79, 103, 127, 151, 167, 191, 199, 223, 239, 263
10n+1: 11, 31, 41, 61, 71, 101, 131, 151, 181, 191, 211, 241, 251, 271, 281
10n+3: 3, 13, 23, 43, 53, 73, 83, 103, 113, 163, 173, 193, 223, 233, 263
10n+7: 7, 17, 37, 47, 67, 97, 107, 127, 137, 157, 167, 197, 227, 257, 277
10n+9: 19, 29, 59, 79, 89, 109, 139, 149, 179, 199, 229, 239, 269, 349, 359
10n+d (d = 1, 3, 7, 9) are primes: ending in the decimal digit d.

Right-truncatable primes:

Primes that remain prime when the last decimal digit is successively removed.
2, 3, 5, 7, 23, 29, 31, 37, 53, 59, 71, 73, 79, 233, 239, 293, 311, 313, 317, 373, 379, 593, 599, 719, 733, 739, 797, 2333, 2339, 2393, 2399, 2939, 3119, 3137, 3733, 3739, 3793, 3797

Safe primes:

Where p and (p−1) / 2 are both prime.
5, 7, 11, 23, 47, 59, 83, 107, 167, 179, 227, 263, 347, 359, 383, 467, 479, 503, 563, 587, 719, 839, 863, 887, 983, 1019, 1187, 1283, 1307, 1319, 1367, 1439, 1487, 1523, 1619, 1823, 1907

Self primes in base 10:

Primes that cannot be generated by any integer added to the sum of its decimal digits.
3, 5, 7, 31, 53, 97, 211, 233, 277, 367, 389, 457, 479, 547, 569, 613, 659, 727, 839, 883, 929, 1021, 1087, 1109, 1223, 1289, 1447, 1559, 1627, 1693, 1783, 1873

Sexy primes:

Where (p, p+6) are both prime.
(5, 11), (7, 13), (11, 17), (13, 19), (17, 23), (23, 29), (31, 37), (37, 43), (41, 47), (47, 53), (53, 59), (61, 67), (67, 73), (73, 79), (83, 89), (97, 103), (101, 107), (103, 109), (107, 113), (131, 137), (151, 157), (157, 163), (167, 173), (173, 179), (191, 197), (193, 199)

Smarandache–Wellin primes:

Primes which are the concatenation of the first n primes: written in decimal.
2, 23, 2357
The fourth Smarandache-Wellin prime is the 355-digit concatenation of the first 128 primes: which end with 719.

Solinas primes:

Of the form $2^a \pm 2^b \pm 1$, where $0 < b < a$.
3, 5, 7, 11, 13

Sophie Germain primes:

Where p and 2p+1 are both prime.
2, 3, 5, 11, 23, 29, 41, 53, 83, 89, 113, 131, 173, 179, 191, 233, 239, 251, 281, 293, 359, 419, 431, 443, 491, 509, 593, 641, 653, 659, 683, 719, 743, 761, 809, 911, 953

Star primes:

Of the form 6n(n − 1) + 1.
13, 37, 73, 181, 337, 433, 541, 661, 937, 1093, 2053, 2281, 2521, 3037, 3313, 5581, 5953, 6337, 6733, 7561, 7993, 8893, 10333, 10837, 11353, 12421, 12973, 13537, 15913, 18481

Stern primes:
Primes that are not the sum of a smaller prime and twice the square of a nonzero integer.
2, 3, 17, 137, 227, 977, 1187, 1493

Super-primes:
Primes with a prime index in the sequence of prime numbers (the 2nd, 3rd, 5th, ... prime).
3, 5, 11, 17, 31, 41, 59, 67, 83, 109, 127, 157, 179, 191, 211, 241, 277, 283, 331, 353, 367, 401, 431, 461, 509, 547, 563, 587, 599, 617, 709, 739, 773, 797, 859, 877, 919, 967, 991

Supersingular primes:
2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 41, 47, 59, 71

Swinging primes:
Primes which are within 1 of a swinging factorial: $n\wr \pm 1$.
2, 3, 5, 7, 19, 29, 31, 71, 139, 251, 631, 3433, 12011

Thabit number primes:
Of the form $3\times 2^n - 1$.
2, 5, 11, 23, 47, 191, 383, 6143, 786431, 51539607551, 824633720831, 26388279066623, 108086391056891903, 55340232221128654847, 226673591177742970257407

Prime triplets:
Where (p, p+2, p+6) or (p, p+4, p+6) are all prime.
(5, 7, 11), (7, 11, 13), (11, 13, 17), (13, 17, 19), (17, 19, 23), (37, 41, 43), (41, 43, 47), (67, 71, 73), (97, 101, 103), (101, 103, 107), (103, 107, 109), (107, 109, 113), (191, 193, 197), (193, 197, 199), (223, 227, 229), (227, 229, 233), (277, 281, 283), (307, 311, 313), (311, 313, 317), (347, 349, 353)

Twin primes:
Where (p, p+2) are both prime.
(3, 5), (5, 7), (11, 13), (17, 19), (29, 31), (41, 43), (59, 61), (71, 73), (101, 103), (107, 109), (137, 139), (149, 151), (179, 181), (191, 193), (197, 199), (227, 229), (239, 241), (269, 271), (281, 283), (311, 313), (347, 349), (419, 421), (431, 433), (461, 463)

Two-sided primes:
Primes which are both left-truncatable and right-truncatable. There are exactly fifteen two-sided primes::
2, 3, 5, 7, 23, 37, 53, 73, 313, 317, 373, 797, 3137, 3797, 739397

Ulam number primes:
Ulam numbers that are prime.
2, 3, 11, 13, 47, 53, 97, 131, 197, 241, 409, 431, 607, 673, 739, 751, 983, 991, 1103, 1433, 1489, 1531, 1553, 1709, 1721, 2371, 2393, 2447, 2633, 2789, 2833, 2897

Unique primes:
The list of primes p for which the period length of the decimal expansion of 1/p is unique (no other prime gives the same period).
3, 11, 37, 101, 9091, 9901, 333667, 909091, 99990001, 999999000001, 9999999900000001, 909090909090909091, 1111111111111111111, 11111111111111111111111, 900900900900990990990991

Wagstaff primes:
Of the form $(2^n+1) / 3$.
3, 11, 43, 683, 2731, 43691, 174763, 2796203, 715827883, 2932031007403, 768614336404564651, 201487636602438195784363, 845100400152152934331135470251, 56713727820156410577229101238628035243

Wall-Sun-Sun primes:

A prime $p > 5$ if p^2 divides the Fibonacci number $F_{p-\left(\frac{p}{5}\right)}$, where the Legendre symbol $\left(\frac{p}{5}\right)$ is defined as

$$\left(\frac{p}{5}\right) = \begin{cases} 1 & \text{if } p \equiv \pm 1 \pmod 5 \\ -1 & \text{if } p \equiv \pm 2 \pmod 5 \end{cases}$$ As of 2011, no Wall-Sun-Sun primes: are known.

Wedderburn-Etherington number primes:
Wedderburn-Etherington numbers that are prime.
2, 3, 11, 23, 983, 2179, 24631, 3626149, 253450711, 596572387

Weakly prime numbers
Primes that having any one of their (base 10) digits changed to any other value will always result in a composite number.
294001, 505447, 584141, 604171, 971767, 1062599, 1282529, 1524181, 2017963, 2474431, 2690201, 3085553, 3326489, 4393139

Wieferich primes:
Primes p for which p^2 divides $2^{p-1} - 1$.
1093, 3511

Wieferich primes: base 3 (Mirimanoff primes:)
Primes p for which p^2 divides $3^{p-1} - 1$.
11, 1006003

Wieferich primes: base 5
Primes p for which p^2 divides $5^{p-1} - 1$
2, 20771, 40487, 53471161, 1645333507, 6692367337, 188748146801

Wieferich primes: base 6
Primes p for which p^2 divides $6^{p-1} - 1$.
66161, 534851, 3152573

Wieferich primes: base 7
Primes p for which p^2 divides $7^{p-1} - 1$.
5, 491531

Wieferich primes: base 10
Primes p for which p^2 divides $10^{p-1} - 1$.
3, 487, 56598313

Wieferich primes: base 11
Primes p for which p^2 divides $11^{p-1} - 1$.
71

Wieferich primes: base 12
Primes p for which p^2 divides $12^{p-1} - 1$.
2693, 123653

Wieferich primes: base 13
Primes p for which p^2 divides $13^{p-1} - 1$
863, 1747591

Wieferich primes: base 17
Primes p for which p^2 divides $17^{p-1} - 1$.
3, 46021, 48947

Wieferich primes: base 19
Primes p for which p^2 divides $19^{p-1} - 1$
3, 7, 13, 43, 137, 63061489

Wilson primes:
Primes p for which p^2 divides $(p-1)! + 1$.
5, 13, 563

Wolstenholme primes:

Primes p for which the binomial coefficient $\binom{2p-1}{p-1} \equiv 1 \pmod{p^4}$
16843, 2124679

Woodall primes:
Of the form $n \times 2^n - 1$.
7, 23, 383, 32212254719, 2833419889721787128217599, 195845982777569926302400511, 4776913109852041418248056622882488319

4.13 GENERALISATIONS FROM PRIME NUMBERS:

Perfect Numbers: A perfect number is a positive integer that is equal to the sum of its proper positive divisors, excluding the number itself. Even perfect numbers are of the form $2^{p-1}(2^p-1)$, where (2^p-1) is prime and by extension p is also prime. It is unknown whether there are any odd perfect numbers.

List of Perfect Numbers:

Rank	*p*	Perfect number	Digits	Year	Discoverer
1	2	6	1	Known to the Greeks	
2	3	28	2	Known to the Greeks	
3	5	496	3	Known to the Greeks	
4	7	8128	4	Known to the Greeks	
5	13	33550336	8	1456	First seen in the medieval manuscript, Codex Lat. Monac.
6	17	8589869056	10	1588	Cataldi
7	19		12	1588	Cataldi
8	31		19	1772	Euler
9	61		37	1883	Pervushin
10	89		54	1911	Powers
11	107		65	1914	Powers
12	127		77	1876	Lucas
13	521		314	1952	Robinson
14	607		366	1952	Robinson
15	1279		770	1952	Robinson
16	2203		1327	1952	Robinson
17	2281		1373	1952	Robinson
18	3217		1937	1957	Riesel
19	4253		2561	1961	Hurwitz
20	4423		2663	1961	Hurwitz
21	9689		5834	1963	Gillies
22	9941		5985	1963	Gillies

23	11213		6751	1963	Gillies
24	19937		12003	1971	Tuckerman
25	21701		13066	1978	Noll & Nickel
26	23209		13973	1979	Noll
27	44497		26790	1979	Nelson & Slowinski
28	86243		51924	1982	Slowinski
29	110503		66530	1988	Colquitt & Welsh
30	132049		79502	1983	Slowinski
31	216091		130100	1985	Slowinski
32	756839		455663	1992	Slowinski & Gage
33	859433		517430	1994	Slowinski & Gage
34	1257787		757263	1996	Slowinski & Gage
35	1398269		841842	1996	Armengaud, Woltman, et al.
36	2976221		1791864	1997	Spence, Woltman, et al.
37	3021377		1819050	1998	Clarkson, Woltman, Kurowski, et al.
38	6972593		4197919	1999	Hajratwala, Woltman, Kurowski, et al.
39	13466917		8107892	2001	Cameron, Woltman, Kurowski, et al.
40	20996011		12640858	2003	Shafer, Woltman, Kurowski, et al.
41	24036583		14471465	2004	Findley, Woltman, Kurowski, et al.
42	25964951		15632458	2005	Nowak, Woltman, Kurowski, et al.
43	30402457		18304103	2005	Cooper, Boone, Woltman, Kurowski, et al.
44	32582657		19616714	2006	Cooper, Boone, Woltman, Kurowski, et al.
45	37156667		22370543	2008	Elvenich, Woltman, Kurowski, et al.
46	42643801		25674127	2009	Strindmo, Woltman, Kurowski, et al.
47	43112609		25956377	2008	Smith, Woltman, Kurowski, et al.

Amicable Numbers: Amicable numbers are two different numbers so related that the sum of the proper divisors of each is equal to the other number.

List of Amicable Numbers:

Amicable Pairs		Amicable Pairs		Amicable Pairs	
220	284	1,328,470	1,483,850	8,619,765	9,627,915
1,184	1,210	1,358,595	1,486,845	8,666,860	10,638,356
2,620	2,924	1,392,368	1,464,592	8,754,130	10,893,230
5,020	5,564	1,466,150	1,747,930	8,826,070	10,043,690
6,232	6,368	1,468,324	1,749,212	9,071,685	9,498,555
10,744	10,856	1,511,930	1,598,470	9,199,496	9,592,504
12,285	14,595	1,669,910	2,062,570	9,206,925	10,791,795
17,296	18,416	1,798,875	1,870,245	9,339,704	9,892,936
63,020	76,084	2,082,464	2,090,656	9,363,584	9,437,056
66,928	66,992	2,236,570	2,429,030	9,478,910	11,049,730
67,095	71,145	2,652,728	2,941,672	9,491,625	10,950,615
69,615	87,633	2,723,792	2,874,064	9,660,950	10,025,290
79,750	88,730	2,728,726	3,077,354	9,773,505	11,791,935
100,485	124,155	2,739,704	2,928,136	10,254,970	10,273,670
122,265	139,815	2,802,416	2,947,216	10,533,296	10,949,704
122,368	123,152	2,803,580	3,716,164	10,572,550	10,854,650
141,664	153,176	3,276,856	3,721,544	10,596,368	11,199,112
142,310	168,730	3,606,850	3,892,670	10,634,085	14,084,763
171,856	176,336	3,786,904	4,300,136	10,992,735	12,070,305

176,272	180,848	3,805,264	4,006,736	11,173,460	13,212,076
185,368	203,432	4,238,984	4,314,616	11,252,648	12,101,272
196,724	202,444	4,246,130	4,488,910	11,498,355	12,024,045
280,540	365,084	4,259,750	4,445,050	11,545,616	12,247,504
308,620	389,924	4,482,765	5,120,595	11,693,290	12,361,622
319,550	430,402	4,532,710	6,135,962	11,905,504	13,337,336
356,408	399,592	4,604,776	5,162,744	12,397,552	13,136,528
437,456	455,344	5,123,090	5,504,110	12,707,704	14,236,136
469,028	486,178	5,147,032	5,843,048	13,671,735	15,877,065
503,056	514,736	5,232,010	5,799,542	13,813,150	14,310,050
522,405	525,915	5,357,625	5,684,679	13,921,528	13,985,672
600,392	669,688	5,385,310	5,812,130	14,311,688	14,718,712
609,928	686,072	5,459,176	5,495,264	14,426,230	18,087,818
624,184	691,256	5,726,072	6,369,928	14,443,730	15,882,670
635,624	712,216	5,730,615	6,088,905	14,654,150	16,817,050
643,336	652,664	5,864,660	7,489,324	15,002,464	15,334,304
667,964	783,556	6,329,416	6,371,384	15,363,832	16,517,768
726,104	796,696	6,377,175	6,680,025	15,938,055	17,308,665
802,725	863,835	6,955,216	7,418,864	16,137,628	16,150,628
879,712	901,424	6,993,610	7,158,710	16,871,582	19,325,698
898,216	980,984	7,275,532	7,471,508	17,041,010	19,150,222
947,835	1,125,765	7,288,930	8,221,598	17,257,695	17,578,785
998,104	1,043,096	7,489,112	7,674,088	17,754,165	19,985,355
1,077,890	1,099,390	7,577,350	8,493,050	17,844,255	19,895,265
1,154,450	1,189,150	7,677,248	7,684,672	17,908,064	18,017,056
1,156,870	1,292,570	7,800,544	7,916,696	18,056,312	18,166,888
1,175,265	1,438,983	7,850,512	8,052,488	18,194,715	22,240,485
1,185,376	1,286,744	8,262,136	8,369,864	18,655,744	19,154,336
1,280,565	1,340,235				

Sociable Numbers: Sociable numbers are generalisations of amicable numbers where a sequence of numbers each of whose numbers is the sum of the factors of the preceding number, excluding the preceding number itself. The sequence must be cyclic, eventually returning to its starting point

.

List of Sociable Numbers:

C4s	
1264460	
1547860	
1727636	
1305184	
2115324	
3317740	
3649556	
2797612	
2784580	
3265940	
3707572	

3370604	
4938136	
5753864	
5504056	
5423384	
7169104	
7538660	
8292568	
7520432	
C5 Poulet 1918 5D	
12496	2^4*11*71
14288	2^4*19*47
15472	2^4*967
14536	2^3*23*79
14264	2^3*1783
C6 Moews&Moews 1992 11D	
21548919483	3^5*7^2*13*19*17*431
23625285957	3^5*7^2*13*19*29*277
24825443643	3^2*7^2*13*19*11*20719
26762383557	3^4*7^2*13*19*27299
25958284443	3^2*7^2*13*19*167*1427
23816997477	3^2*7^2*13*19*218651
C6 Moews&Moews 1995 11D/12D	
90632826380	2^2*5*109*431*96461
101889891700	2^2*5^2*31*193*170299
127527369100	2^2*5^2*31*181*227281
159713440756	2^2*31*991*1299709
129092518924	2^2*31*109*9551089
106246338676	2^2*17*25411*61487
C6 Needham 2006 13D	
1771417411016	2^3*11*20129743307
1851936384424	2^3*7*1637*20201767
2118923133656	2^3*7*863*43844627
2426887897384	2^3*59*5141711647
2200652585816	2^3*43*1433*4464233
2024477041144	2^3*253059630143
C6 Needham 2006 13D	
3524434872392	2^3*7*17*719*5149009
4483305479608	2^3*89*6296777359
4017343956392	2^3*13*17*3019*752651
4574630214808	2^3*607*6779*138967
4018261509992	2^3*31*59*274621481
3890837171608	2^3*61*22039*361769

C6 Needham 2006 13D	
4773123705616	2^4*7*347*122816069
5826394399664	2^4*101*3605442079
5574013457296	2^4*53*677*1483*6547
5454772780208	2^4*53*239*2971*9059
5363145542992	2^4*307*353*3093047
5091331952624	2^4*318208247039
C8 Flammenkamp 1990 Brodie ? 10D	
1095447416	2^3*7*313*62497
1259477224	2^3*43*3661271
1156962296	2^3*7*311*66431
1330251784	2^3*43*3867011
1221976136	2^3*41*1399*2663
1127671864	2^3*11*61*83*2531
1245926216	2^3*19*8196883
1213138984	2^3*67*2263319
C8 Flammenkamp 1990 Brodie ? 10D	
1276254780	2^2*3*5*1973*10781
2299401444	2^2*3*991*193357
3071310364	2^2*767827591
2303482780	2^2*5*67*211*8147
2629903076	2^2*23*131*218213
2209210588	2^2*13^2*17*192239
2223459332	2^2*131*4243243
1697298124	2^2*907*467833
C9 Flammenkamp 1990 9D/10D	
805984760	2^3*5*7*1579*1823
1268997640	2^3*5*17*61*30593
1803863720	2^3*5*103*367*1193
2308845400	2^3*5^2*11544227
3059220620	2^2*5*2347*65173
3367978564	2^2*841994641
2525983930	2*5*17*367*40487
2301481286	2*13*19*4658869
1611969514	2*805984757
C28 Poulet 1918 5D/6D	
14316	2^2*3*1193
19116	2^2*3^4*59
31704	2^3*3*1321
47616	2^9*3*31
83328	2^7*3*7*31
177792	2^7*3*463
295488	2^6*3^5*19
629072	2^4*39317
589786	2*294893
294896	2^4*7*2633

358336	2^6*11*509
418904	2^3*52363
366556	2^2*91639
274924	2^2*13*17*311
275444	2^2*13*5297
243760	2^4*5*11*277
376736	2^5*61*193
381028	2^2*95257
285778	2*43*3323
152990	2*5*15299
122410	2*5*12241
97946	2*48973
48976	2^4*3061
45946	2*22973
22976	2^6*359
22744	2^3*2843
19916	2^2*13*383
17716	2^2*43*103
This list is exhaustive for known social numbers where C>4	

4.14 GOLDEN RATIO & FIBONACCI SEQUENCE:

Relationship:

$$\frac{a+b}{a} = \frac{a}{b} \equiv \varphi,$$

$$\frac{a+b}{a} = 1 + \frac{b}{a} = 1 + \frac{1}{\varphi},$$

$$\varphi = \frac{1+\sqrt{5}}{2}$$

$$\varphi = 1.6180339887498948482045868343656381177203091798058$$

$$\varphi^{n+1} = \varphi^n + \varphi^{n-1}.$$

Infinite Series:

$$\varphi = \sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\cdots}}}}.$$

$$\varphi = \frac{13}{8} + \sum_{n=0}^{\infty} \frac{(-1)^{(n+1)}(2n+1)!}{(n+2)!n!4^{(2n+3)}}.$$

Continued Fractions:

$$\varphi = [1;1,1,1,\dots] = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \ddots}}}$$

$$\varphi^{-1} = [0; 1, 1, 1, \dots] = 0 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \ddots}}}$$

Trigonometric Expressions:

$$\varphi = 1 + 2\sin(\pi/10) = 1 + 2\sin 18^\circ$$
$$\varphi = \frac{1}{2}\csc(\pi/10) = \frac{1}{2}\csc 18^\circ$$
$$\varphi = 2\cos(\pi/5) = 2\cos 36^\circ$$
$$\varphi = 2\sin(3\pi/10) = 2\sin 54^\circ.$$

Fibonacci Sequence:

$$F(n) = \frac{\varphi^n - (1-\varphi)^n}{\sqrt{5}} = \frac{\varphi^n - (-\varphi)^{-n}}{\sqrt{5}}$$

$$F(n) = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right)$$

$$\lim_{n\to\infty}\frac{F(n+1)}{F(n)} = \varphi.$$

$$\sum_{n=1}^{\infty}|F(n)\varphi - F(n+1)| = \varphi.$$

$$\lim_{n\to\infty}\frac{F(n+a)}{F(n)} = \varphi^a,$$

4.15 FERMAT'S LAST THEOREM:

$a^n + b^n \neq c^n$ for integers $a, b \,\&\, c$ and $n > 2$

Proposed by Fermat in 1637 as an extension of Diophantus's explanation of the case when n=2.
Case when n=3 was proved by Euler (1770)
Case when n=4 was proved by Fermat
Case when n=5 was proved by Legendre and Dirichlet (1885)
Case when n=7 was proved by Gabriel Lamé (1840)
General case when n>2 was proved by Andrew Wiles (1994). The proof is too long to be written here. See: http://www.cs.berkeley.edu/~anindya/fermat.pdf

4.16 BOOLEAN ALGEBRA:

Axioms:

Axiom	Dual	Name
$B = 0$ if $B \neq 1$	$B = 1$ if $B \neq 0$	Binary Field
$\bar{0} = 1$	$\bar{1} = 0$	NOT
$0 \bullet 0 = 0$	$1 + 1 = 1$	AND/OR
$1 \bullet 1 = 1$	$0 + 0 = 0$	AND/OR
$0 \bullet 1 = 1 \bullet 0 = 0$	$0 + 1 = 1 + 0 = 1$	AND/OR

Theorems of one variable:

Theorem	Dual	Name
$B \bullet 1 = B$	$B + 0 = B$	Identity
$B \bullet 0 = 0$	$B + 1 = 1$	Null Element
$B \bullet B = B$	$B + B = B$	Idempotency
$\overline{\overline{B}} = B$		Involution
$B \bullet \overline{B} = 0$	$B + \overline{B} = 1$	Complements

Theorems of several variables:

Theorem	Dual	Name
$B \bullet C = C \bullet B$	$B + C = C + B$	Commutativity
$(B \bullet C) \bullet D = B \bullet (C \bullet D)$	$(B + C) + D = B + (C + D)$	Associativity
$(B \bullet C) + (B \bullet D) = B \bullet (C + D)$	$(B + C) \bullet (B + D) = B + (C \bullet D)$	Distributivity
$B \bullet (B + C) = B$	$B + (B \bullet C) = B$	Covering
$(B \bullet C) + (B \bullet \overline{C}) = B$	$(B + C) \bullet (B + \overline{C}) = B$	Combining
$(B \bullet C) + (B \bullet D) + (C \bullet D) = B \bullet C + \overline{B} \bullet D$	$(B \bullet C) + (B \bullet D) + (C \bullet D) = B \bullet C$	Consesnus
$\overline{B_1 \bullet B_2 \bullet \ldots} = (\overline{B_1} + \overline{B_2} + \ldots)$	$\overline{B_1 + B_2 + \ldots} = (\overline{B_1} \bullet \overline{B_2} \bullet \ldots)$	De Morgan’s

PART 5: COUNTING TECHNIQUES & PROBABILITY

5.1 2D

Triangle Number $T_n = \frac{n(n+1)}{2}$

$n^2 = T_n + T_{n-1}$

Square Number $T_n = n^2$

Pentagonal Number $T_n = \frac{n(3n-1)}{2}$

5.2 3D

Tetrahedral Number $T_n = \frac{n^3 + 3n^2 + 2n}{6}$

Square Pyramid Number $T_n = \frac{2n^3 + 3n^2 + n}{6}$

5.3 PERMUTATIONS

Permutations: $= n!$

Permutations (with repeats): $= \frac{n!}{(groupA)! \times (groupB)! \times \ldots}$

5.4 COMBINATIONS

Ordered Combinations: $= \frac{n!}{(n-p)!}$

Unordered Combinations: $= \binom{n}{p} = \frac{n!}{p!(n-p)!}$

Ordered Repeated Combinations: $= n^p$

Unordered Repeated Combinations: $= \frac{(p+n-1)!}{p! \times (n-1)!}$

Grouping: $= \binom{n}{n_1}\binom{n-n_1}{n_2}\binom{n-n_1-n_2}{n_3}\ldots = \frac{n!}{n_1!n_2!n_3!\ldots n_r!}$

5.5 MISCELLANEOUS:

Total Number of Rectangles and Squares from a a x b rectangle:

$\sum = T_a \times T_b$

Number of Interpreters: $= T_{L-1}$

Max number of pizza pieces: $= \frac{c(c+1)}{2} + 1$

Max pieces of a crescent: $= \frac{c(c+3)}{2} + 1$

Max pieces of cheese: $= \frac{c^3 + 5c}{6} + 1$

Cards in a card house: $= \frac{l(3l+1)}{2}$

Different arrangement of dominos: $= 2^{n-d} \times n!$

Unit Fractions: $\frac{a}{b} = \frac{1}{INT\left[\frac{b}{a}\right]+1} + \frac{a - MOD\left[\frac{b}{a}\right]}{b\left(INT\left[\frac{b}{a}\right]+1\right)}$

Angle between two hands of a clock: $\theta = 5.5m - 30h$

Winning Lines in Noughts and Crosses: $= 2(a+1)$

Bad Restaurant Spread: $= \frac{P}{1-s}$

Fibonacci Sequence: $= \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right)$

ABBREVIATIONS (5.1, 5.2, 5.3, 5.4, 5.5)

a=side 'a'
b=side 'b'
c=cuts
d=double dominos
h=hours
L=Languages
l=layers
m=minutes
n= nth term
n=n number
P=Premium/Starting Quantity
p=number you pick
r=number of roles/turns
s=spread factor
T=Term
θ=the angle

5.6 FACTORIAL:

Definition: $n! = n \times (n-1) \times (n-2) \times ... \times 2 \times 1$

Table of Factorials:

0!	1 (by definition)		
1!	1	11!	39916800
2!	2	12!	479001600
3!	6	13!	6227020800
4!	24	14!	87178291200
5!	120	15!	1307674368000
6!	720	16!	20922789888000
7!	5040	17!	355687428096000

8!	40320	18!	6402373705728000
9!	362880	19!	121645100408832000
10!	3628800	20!	2432902008176640000

Approximation: $n! \approx \sqrt{2\pi} \times n^{n+\frac{1}{2}} \times e^{-n}$ (within 1% for n>10)

5.7 THE DAY OF THE WEEK:

This only works after 1753

$$= MOD7\left(d + y + \left[\frac{31m}{12}\right] + \left[\frac{y}{4}\right] - \left[\frac{y}{100}\right] + \left[\frac{y}{400}\right]\right)$$

d=day
m=month
y=year
SQUARE BRAKETS MEAN INTEGER DIVISION
INT=Keep the integer
MOD=Keep the remainder

5.8 BASIC PROBABILITY:

Axiom's of Probability:

1. $P(\Omega) = 1$ for the eventspace Ω
2. $P(A) \in [0,1]$ for any event A.
3. If A_1 and A_2 are disjoint, then $P(A_1 \cup A_2) = P(A_1) + P(A_2)$

   Generally, if A_i are mutually disjoint, then $P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i)$

Commutative Laws:

$$A \cup B = B \cup A$$
$$A \cap B = B \cap A$$

Associative Laws:

$$(A \cup B) \cup C = A \cup (B \cup C)$$
$$(A \cap B) \cap C = A \cap (B \cap C)$$

Distributive Laws:

$$(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$$
$$(A \cap B) \cup C = (A \cup C) \cap (B \cup C)$$

Indicator Function:

$$\mathrm{I}_D = \begin{cases} 1 & \text{if point is} \in \text{D} \\ 0 & \text{if point is} \notin \text{D} \end{cases}$$

5.9 VENN DIAGRAMS:

Complementary Events: $1 - P(A) = P(\overline{A})$

Null Set: $P(\Phi) = 0$

Totality:

$$P(A) = \sum_{i=1}^{m} P(A \mid B_i) P(B_i)$$
$$P(A) = P(A \mid B_1) P(B_1) + P(A \mid B_2) P(B_2) + \ldots$$

where $B_i \cap B_j = \Phi$ for $i \neq j$

$$P(A) = P(A \cap B) + P(A \cap B')$$

Conditional Probability: $P(A|B)=\dfrac{P(A\cap B)}{P(B)}$

$P(A\cap B)=P(B)\cdot P(A|B)=P(A)\cdot P(B|A)$ (Multiplication Law)

Union: $P(A\cup B)=P(A)+P(B)-P(A\cap B)$ (Addition Law)

Independent Events: $P(A\cap B)=P(A)\cdot P(B)$

$P(A\cup B)=P(A)+P(B)-P(A)\cdot P(B)$

$P(B|A)=P(B)$

$P(A_1\cap A_2\cap\ldots\cap A_k)=P(A_1)P(A_2)\ldots P(A_k)$

Mutually Exclusive: $P(A\cap B)=0$

$P(A\cap B')=P(A)$

$P(A\cup B)=P(A)+P(B)$

$P(A\cup B')=P(B')$

Subsets: if $A\subset B$ then $P(A)\le P(B)$

Baye's Theorem:

$$P(B|A)=\frac{P(A|B)P(B)}{P(A)}=\frac{P(A|B)P(B)}{P(A|B)P(B)+P(A|B')P(B')}$$

Event's Space: $P(A)=\sum_{i=1}^{m}P(A\cap B_i)$

5.11 BASIC STATISTICAL OPERATIONS:

Variance: $v=\sigma^2$

Arithmetic Mean: $\dfrac{\mu-a}{b-\mu}=\dfrac{a}{a}=\dfrac{b}{b}=\dfrac{\mu}{\mu}\Rightarrow\mu=\dfrac{a+b}{2}$

$$\mu=\frac{\sum x_i}{n_s}$$

Geometric Mean: $\dfrac{b-\mu}{\mu-a}=\dfrac{\mu}{a}=\dfrac{b}{\mu}\Rightarrow\mu=\sqrt{ab}$

Harmonic Mean: $\dfrac{b-\mu}{\mu-a}=\dfrac{b}{a}\Rightarrow\mu=\dfrac{2}{1/a+1/b}$

Standardized Score: $z=\dfrac{x_i-\mu}{\sigma}$

Confidence Interval:

Quantile: The p[th] of quantile of the distribution F is defined to be the value x_p such that

$F(x_p)=p$ or $P(X\le x_p)=p$

$\therefore x_p=F^{-1}(p)$

$x_{0.25}$ is the lower quartile

$x_{0.5}$ is the median

$x_{0.75}$ is the upper quartile

5.12 DISCRETE RANDOM VARIABLES:

Standard Deviation: $\sigma=\sqrt{\dfrac{\sum\left(x_i-\bar{x}\right)^2}{n_s}}$

Expected Value:

$$E[X] = \sum_{1}^{i} P(x_i) \times x_i$$

$$E[aX + b] = aE[X] + b$$

Variance:

$$v = \frac{\sum (x_i - \bar{x})^2}{n_s}$$

$$v = (E[x - E[x]])^2$$

$$v = E[x^2] - (E[x])^2$$

$$\text{var}[aX + b] = a^2 \text{ var}[X]$$

Probability Mass Function: $P(x) = f(x) = P(X = x)$

Cumulative Distribution Function: $F(x) = P(X \le x)$

$$F(x + 0) = F(x)$$

5.13 COMMON DRVs:

Bernoulli Trial:

Definition:	1 trial, 1 probability that is either fail or success
Example:	Probability of getting a 6 from one roll of a die
Outcomes:	$S_X = \{0,1\}$
Probability:	$P_X(x) = \begin{cases} p & x = 1 \\ 1 - p & x = 0 \end{cases}$
Expected Value:	$E[X] = p$
Variance:	$Var[X] = p - p^2 = p(1 - p)$

Binomial Trial:

Definition:	Repeated Bernoulli Trials
Example:	Probability of getting x 6s from n rolls of a die
Outcomes:	$S_X = \{0,1,2,3,...n\}$
Probability:	$P_X(x) = \binom{n}{x} \cdot (p)^x \cdot (1 - p)^{n-x}$
Expected Value:	$E[X] = np$
Variance:	$Var[X] = np(1 - p)$

n=number to choose from
p=probability of x occurring
x=number of favorable results

Poisson Distribution:

Definition:	The limit of the binomial distribution as $n \to \infty, p \to 0$
Example:	Probability of getting x 6s from n rolls of a die
Outcomes:	$S_X = \{0,1,2,3,...n\}$
Probability:	$P_X(x) = \frac{\lambda^x}{x!} e^{-\lambda}, \lambda = np$
Expected Value:	$E[X] = np = \lambda$
Variance:	$Var[X] = np = \lambda$

Geometric Binomial Trial:

Definition: Number of Bernoulli Trials to get 1st Success.

Example: Probability of getting the first 6s from n rolls of a die

Outcomes: $S_X = \{0,1,2,3,...\}$

Probability: $P_X(x) = p(1-p)^{x-1}$

Expected Value: $E[X] = \frac{1}{p}$

Variance: $Var[X] = \frac{1-p}{p^2}$

Negative Binomial Trial:

Definition: Number to 1st get to n successes.

Example: Probability of getting the first n 6s from x rolls of a die

Probability: $P_X(x) = \binom{x-1}{n-1} p^x (1-p)^{n-x}$

Expected Value: $E[X] = \frac{k}{p}$

Variance: $Var[X] = \frac{kn(1-p)}{p^2}$

Hypergeometric Trial:

Definition: Number of choosing k out of m picks from n possiblities without replacement

Example: An urn with black balls and other coloured balls. Finding the probability of getting m black balls out of n without replacement.

Probability: $P_X(x) = \frac{\binom{r}{k}\binom{n-r}{m-k}}{\binom{n}{m}}$

Expected Value: $E[X] = \frac{mr}{n}$

n = number of balls
r = number of black balls
m = number of balls drawn from urn without replacement
k = number of black balls drawn from urn

5.14 CONTINUOUS RANDOM VARIABLES:

Probability Density Function: $= f(x)$

$$\text{If } \int_{-\infty}^{\infty} f(x)dx = 1 \;\&\; f(x) \geq 0 \; for -\infty \leq x \leq \infty$$

Cumulative Distribution Function: $= F(x) = P(X \leq x) = \int_{-\infty}^{x} f(x)dx$

Interval Probability: $P(a \leq X \leq b) = F(b) - F(a) = \int_{a}^{b} f(x)dx$

Expected Value: $$E(x) = \int_{-\infty}^{\infty} x \times f(x)dx$$

$$E(g(x)) = \int_{-\infty}^{\infty} g(x) \times f(x)dx$$

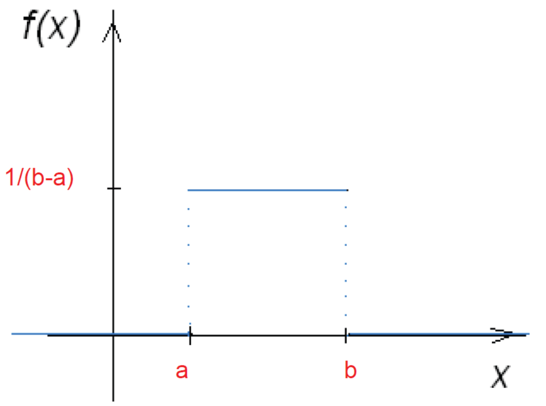
Variance: $$Var(X) = E(X^2) - (E(X))^2$$

5.15 COMMON CRVs:

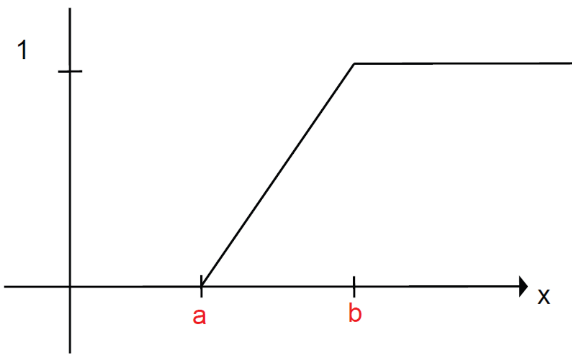
Uniform Distribution:

Declaration: $X \sim Uniform(a,b)$

PDF: $$f(x) = \begin{cases} \dfrac{1}{b-a} & a \le x \le b \\ 0 & otherwise \end{cases}$$



CDF: $$F(x) = \int_{-\infty}^{x} f(x)dx = \begin{cases} 0 & x < a \\ \dfrac{x-a}{b-a} & a \le x \le b \\ 1 & x > b \end{cases}$$



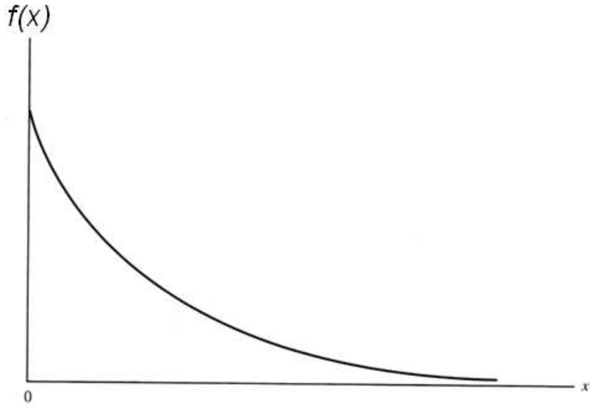
Expected Value: $= \dfrac{a+b}{2}$

Variance: $= \dfrac{(b-a)^2}{12}$

Exponential Distribution:

Declaration: $X \sim Exponential(\lambda)$

PDF: $f(x) = \begin{cases} 0 & x < 0 \\ \lambda e^{-\lambda x} & x \geq 0 \end{cases}$



CDF: $F(x) = \int_{-\infty}^{x} f(x)dx = \begin{cases} 0 & x < 0 \\ 1 - e^{-\lambda x} & x \geq 0 \end{cases}$

Expected Value: $= \frac{1}{\lambda}$

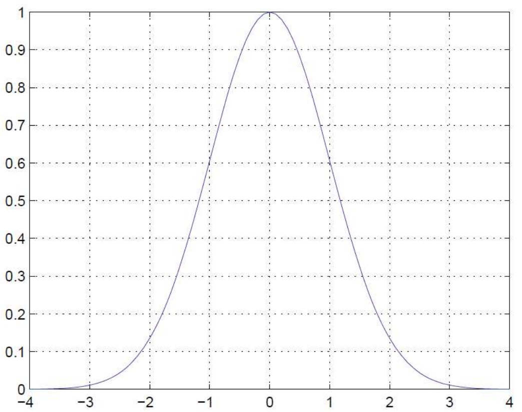
Variance: $= \frac{1}{\lambda^2}$

Normal Distribution:

Declaration: $X \sim Normal(\mu, \sigma^2)$

Standardized Z Score: $Z = \frac{x - \mu}{\sigma}$

PDF: $f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} = \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-1}{2}z^2}$



CDF: $\Phi(Z)$ (The integration is provided within statistic tables)

Expected Value: $= \mu$

Variance: $= \sigma^2$

5.16 BIVARIABLE DISCRETE:

Probability:

$$P(X = x, Y = y) = f(x, y)$$

$$P(X \leq x, Y \leq y) = \sum f(x, y) \text{ over all values of x \& y}$$

Marginal Distribution:

$$f_X(x) = \sum_y f(x_i, y)$$

$$f_Y(y) = \sum_x f(x, y_i)$$

Expected Value:

$$E[X] = \sum_x x \times f_X(x)$$

$$E[Y] = \sum_y y \times f_Y(y)$$

$$E[X,Y] = \sum_x \sum_y x \times y \times f_{X,Y}(x,y)$$

Independence: $f(x,y) = f_X(x) \times f_Y(y)$

Covariance: $Cov = E[X,Y] - E[X] \times E[Y]$

5.17 BIVARIABLE CONTINUOUS:

Conditions:

$$f_{X,Y}(x,y) \geq 0 \;\&\; \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x,y)dxdy = 1$$

Probability:

$$P(X \leq x, Y \leq y) = F_{X,Y}(x,y) = \int_{-\infty}^{y}\int_{-\infty}^{x} f_{X,Y}(x,y)dxdy$$

$$P(X < x) = P(X < x, -\infty < Y < \infty) = \int_{-\infty}^{x} f_X(x)dx$$

$$P(Y < y) = P(-\infty < X < \infty, Y < y) = \int_{-\infty}^{y} f_Y(y)dy$$

Where the domain is:

$$D = \{(x,y) : x \in (a,b), y \in (\alpha(x), \beta(x))\}$$

$$\int_D f_{X,Y}(x,y)dxdy = \int_a^b \int_{\alpha(x)}^{\beta(x)} f_{X,Y}(x,y)dydx$$

Where the domain is:

$$D = \{(x,y) : x \in (\gamma(x), \delta(x)), y \in (c,d)\}$$

$$\int_D f_{X,Y}(x,y)dxdy = \int_c^d \int_{\gamma(x)}^{\delta(x)} f_{X,Y}(x,y)dxdy$$

Where the domain is the event space:

$$\int_a^b \int_{\alpha(x)}^{\beta(x)} f_{X,Y}(x,y)dydx = \int_c^d \int_{\gamma(x)}^{\delta(x)} f_{X,Y}(x,y)dxdy = 1$$

Marginal Distribution:

$f_X(x) = \int_a^b f_{X,Y}(x,y)dy$ where a & b are bounds of y

$$f_Y(y)=\int_a^b f_{X,Y}(x,y)dx \text{ where a \& b are bounds of x}$$

Measure:

$$mes(D)=\int_D f_{X,Y}(x,y)dxdy \text{ where } f_{X,Y}(x,y)=1$$

$$mes(D)=\int_D dxdy \text{ (In this case, mes(D) is the area of D)}$$

Expected Value:

$$E[X]=\int_{-\infty}^{\infty} x\times f_X(x)dx$$

$$E[Y]=\int_{-\infty}^{\infty} y\times f_Y(y)dy$$

$$E[X,Y]=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x\times y\times f_{X,Y}(x,y)dxdy$$

Independence: $f(x,y)=f_X(x)\times f_Y(y)$

Conditional:

$$f_{X|Y}(x|y)=\frac{f_{X,Y}(x,y)}{f_Y(y)}$$

$$f_{X|Y}(x|y)=f_X(x|Y=y)$$

$$P(X\in A|Y=y)=\int_A f_X(x|Y=y)dx$$

Covariance: $Cov=E[X,Y]-E[X]\times E[Y]$

Correlation Coefficient: $\rho_{X,Y}=\dfrac{Cov(X,Y)}{\sigma_X\sigma_Y}$

Bivariate Unifrom Distribution:

$$f_{X,Y}(x,y)=k\times \mathrm{I}_D(x,y)=\begin{cases}k & \text{if (x,y)} \in \text{D}\\ 0 & \text{if (x,y)} \notin \text{D}\end{cases}$$

$$F_{X,Y}(x,y)=\int_{-\infty}^{y}\int_{-\infty}^{x} f_{X,Y}(x,y)dxdy$$

$$k=\frac{1}{\int_D dxdy}=\frac{1}{mes(D)}$$

Multivariate Uniform Distribution:

$$f_{X_1,X_2,\ldots}(x_1,x_2,\ldots)=k\times \mathrm{I}_D(x_1,x_2,\ldots)=\begin{cases}k & \text{if } (x_1,x_2,\ldots)\in \text{D}\\ 0 & \text{if } (x_1,x_2,\ldots)\notin \text{D}\end{cases}$$

$$F_{X_1,X_2,\ldots}(x_1,x_2,\ldots)=\int_D f_{X_1,X_2,\ldots}(x_1,x_2,\ldots)dx_1dx_2\ldots$$

$$k=\frac{1}{mes(D)}$$

Bivariate Normal Distribution:

$$f_{X,Y}(x,y)=\frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}}\times e^{\left(\frac{-1}{2(1-\rho^2)}\left(\frac{(x-\mu_X)^2}{\sigma_X^2}+\frac{(y-\mu_Y)^2}{\sigma_Y^2}-\frac{2\rho(x-\mu_X)(y-\mu_Y)}{\sigma_X\sigma_Y}\right)\right)}$$

where $\mu_X,\mu_Y\in\Re \;\&\; \sigma_X,\sigma_Y>0 \;\&\; \rho\in(-1,1)$

Note that the curves in $\Re^2$ where $f_{X,Y}(x,y)=k$ are circles if $\rho=0$ and elipses if $\rho\neq 0$ provided $\sigma_X=\sigma_Y$. The centre is at the point (μ_X,μ_Y)

5.18 FUNCTIONS OF RANDOM VARIABLES:

Sums (Discrete):

$Z=X+Y$

$Z=z$ iff $X=x, Y=z-x$

$$f_Z(z)=P(Z=z)=\sum_x P(X=x,Y=z-x)=\sum_x f_{X,Y}(x,z-x)$$

If X & Y are independent: (ie: the convolution of $f_X(x)$ & $f_Y(x)$)

$$f_Z(z)=\sum_x f_{X,Y}(x,z-x)=\sum_x f_X(x)f_Y(z-x)$$

Sums (Continuous):

$Z=X+Y$

$Z\le z$ iff (X,Y) is in the region $\{x+y\le z\}$

$$F_Z(z)=P(Z\le z)=\int_{-\infty}^{\infty}\int_{-\infty}^{z-x} f_{X,Y}(x,y)dydx$$

$$f_Z(z)=\frac{dF_Z}{z}(z)=\frac{d}{dz}\int_{-\infty}^{\infty}\int_{-\infty}^{z-x} f_{X,Y}(x,y)dydx=\int_{-\infty}^{\infty}\frac{d}{dz}\left(\int_{-\infty}^{z-x} f_{X,Y}(x,y)dy\right)dx=\int_{-\infty}^{\infty} f_{X,Y}(x,z-x)dx$$

If X & Y are independent: (ie: the convolution of $f_X(x)$ & $f_Y(x)$)

$$f_Z(z)=\int_{-\infty}^{\infty} f_{X,Y}(x,z-x)dx=\int_{-\infty}^{\infty} f_X(x)f_Y(z-x)dx$$

Quotients (Discrete):

$Z=Y/X$

$Z=z$ iff $X=x, Y=zx$

$$f_Z(z)=P(Z=z)=\sum_x P(X=x,Y=zx)=\sum_x f_{X,Y}(x,zx)$$

If X & Y are independent:

$$f_Z(z)=\sum_x f_{X,Y}(x,z-x)=\sum_x f_X(x)f_Y(zx)$$

Quotients (Continuous):

$Z=Y/X$

$Z\le z$ iff (X,Y) is in the region $\{y/x\le z\}$

If $x>0$, then $y<zx$. If $x<0$, then $y>zx$. Thus,

$$F_Z(z)=P(Z\le z)=\int_{-\infty}^{0}\int_{zx}^{\infty} f_{X,Y}(x,y)dydx+\int_{0}^{\infty}\int_{-\infty}^{zx} f_{X,Y}(x,y)dydx$$

$$f_Z(z)=\frac{dF_Z}{z}(z)=\frac{d}{dz}\int_{-\infty}^{0}\int_{zx}^{\infty}f_{X,Y}(x,y)dydx+\frac{d}{dz}\int_{0}^{\infty}\int_{-\infty}^{zx}f_{X,Y}(x,y)dydx$$

$$=\int_{-\infty}^{0}\frac{d}{dz}\left(\int_{zx}^{\infty}f_{X,Y}(x,y)dy\right)dx+\int_{0}^{\infty}\frac{d}{dz}\left(\int_{-\infty}^{zx}f_{X,Y}(x,y)dy\right)dx=\int_{-\infty}^{0}-xf_{X,Y}(x,zx)dx+\int_{0}^{\infty}xf_{X,Y}(x,zx)dx$$

$$f_Z(z)=\int_{-\infty}^{\infty}|x|f_{X,Y}(x,zx)dx$$

If X & Y are independent:

$$f_Z(z)=\int_{-\infty}^{\infty}|x|f_X(x)f_Y(zx)dx$$

Maximum:

Assuming that $X_1, X_2,..., X_n$ are independant random variables with cdf F and density f.

$Z=\max(X_1, X_2,..., X_n)$

$Z\le z$ iff $X_i\le z,\forall i$

$$F_Z(z)=P(Z\le z)=P(X_1\le z, X_2\le z,..., X_n\le z)=P(X_1\le z)\times P(X_2\le z)\times...\times P(X_n\le z)=(F_X(z))^n$$

$$f_Z(z)=\frac{d}{dz}(F_X(z))^n=nF_X(z)^{n-1}f_X(z)$$

Minimum:

Assuming that $X_1, X_2,..., X_n$ are independant random variables with cdf F and density f.

$Z=\min(X_1, X_2,..., X_n)$

$Z>z$ iff $X_i>z,\forall i$

$$F_Z(z)=P(Z\le z)=1-P(Z>z)=1-P(X_1>z, X_2>z,..., X_n>z)$$

$$=1-P(X_1>z)\times P(X_2>z)\times...\times P(X_n>z)=1-(1-F_X(z))^n$$

$$f_Z(z)=\frac{d}{dz}\left(1-(1-F_X(z))^n\right)=n(1-F_X(z))^{n-1}f_X(z)$$

Order Statistics:

Assuming that $X_1, X_2,..., X_n$ are independant random variables with cdf F and density f.

Sorting X_i in non decreasing order:

$X_{(1)}\le X_{(2)}\le...\le X_{(n)}$

The k[th] order statistic of a statistical sample is equal to its k[th] smallest value.

Particularly: $X_{(1)}=\min(X_1, X_2,..., X_n)$ and $X_{(n)}=\max(X_1, X_2,..., X_n)$

Let $1<k<n$. The event $x<X_{(k)}\le x+dx$ occurs if:

1. $k-1$ observations are less than x
2. one observation is in the interval $(x, x+dx]$
3. $n-k$ observations are greater than $x+dx$

The probability of any particular arrangment of this type is:

$$=f(x)F(x)^{k-1}(1-F(x))^{n-k}dx$$

By the combination law:

$$f_K(k)=\frac{n!}{(k-1)!(n-k)!}f(x)F(x)^{k-1}(1-F(x))^{n-k}$$

5.19 TRANSFORMATION OF THE JOINT DENSITY:

Bivariate Functions:

Let X and Y have joint density $f_{X,Y}(x,y)$

Let $g_1 : \Re^2 \to \Re$ & $g_2 : \Re^2 \to \Re$

$U = g_1(X,Y)$ & $V = g_2(X,Y)$

Assuming that g_1 & g_2 can be inverted. There exist $h_1 : \Re^2 \to \Re$ & $h_2 : \Re^2 \to \Re$ so that

$X = h_1(U,V)$ & $Y = h_2(U,V)$

Multivariate Functions:

Let $X_1,..., X_n$ have joint density $f_X(x_1,.., x_n)$

Let $g_i : \Re^n \to \Re$

$Y_i = g_i(X_1,..., X_n)$

Assuming that g_i can be inverted. There exist $h_i : \Re^n \to \Re$ so that

$X_i = h_i(Y_1,..., Y_n)$

Jacobian:

$$J(x,y) = \left\| \begin{bmatrix} \frac{\partial g_1}{\partial x}(x,y) & \frac{\partial g_1}{\partial y}(x,y) \\ \frac{\partial g_2}{\partial x}(x,y) & \frac{\partial g_2}{\partial y}(x,y) \end{bmatrix} \right\| = \frac{\partial g_1}{\partial x}(x,y) \times \frac{\partial g_2}{\partial y}(x,y) - \frac{\partial g_1}{\partial y}(x,y) \times \frac{\partial g_2}{\partial x}(x,y)$$

$$J(x,y) = \left\| \begin{bmatrix} \frac{\partial g_1}{x_1}(x) & ... & \frac{\partial g_1}{x_n}(x) \\ ... & ... & ... \\ \frac{\partial g_n}{x_1}(x) & ... & \frac{\partial g_n}{x_n}(x) \end{bmatrix} \right\|$$

There is an assumption that the derivateves exist and that $J(x,y) \neq 0 \forall x, y$

Joint Density:

$$f_{U,V}(u,v) = \frac{f_{X,Y}(h_1(u,v), h_2(u,v))}{|J(h_1(u,v), h_2(u,v))|}$$

$$f_Y(y_1,..., y_n) = \frac{f_X(h_1(x_1,..., x_n),..., h_n(x_1,..., x_n))}{|J(h_1(x_1,..., x_n),..., h_n(x_1,..., x_n))|}$$

ABBREVIATIONS

σ = Standard Deviation

μ = mean

n_s = number of scores

p = probability of favourable result

v = variance

x_i = Individual x score

$\bar{x}$ = mean of the x scores

z = Standardized Score

PART 6: STATISTICAL ANALYSIS

6.1 GENERAL PRINCIPLES:

$$y_{..} = \sum_{i=1} \sum_{j=1} y_{ij}, \overline{y_{..}} = \frac{y_{..}}{N}$$

$$y_{i.} = \sum_{j=1} y_{ij}, \overline{y_{i.}} = \frac{y_{i.}}{n_1}$$

$$y_{.j} = \sum_{i=1} y_{ij}, \overline{y_{.j}} = \frac{y_{.j}}{a}$$

Mean Square Value of x: $MS_x = \frac{SS_x}{df(\text{x})}$

F-Statistic of x: $F = \frac{MS_x}{MSE}$

F-Statistic of the Null Hypothesis: $F_0 = \frac{SS_{TREATMENT} \div (a-1)}{SS_{ERROR} \div (N-a)} \sim F_{a-1,N-a}$

P-Value: $p = P(F > F_0)$

Relative Efficiency: Multiplication of CRD observations that need to be carried out to ignore the effect of a block (or similar).

6.2 CONTINUOUS REPLICATE DESIGN (CRD):

Populations	Observations				Total	Mean
1	y_{11}	y_{12}	…………	y_{1n1}	$y_{1.}$	$\overline{y_{1.}}$
2	y_{21}	y_{22}	…………	y_{2n2}	$y_{2.}$	$\overline{y_{2.}}$
a	y_{a1}	y_{a2}	…………	y_{ana}	$y_{a.}$	$\overline{y_{i.}}$

Treatments: = a
Factors: = 1
Replications per treatment: = n_i
Total Treatments: $= N = n_T = n_1 + n_2 + ... + n_t$

Mathematical Model:

$$y_{ij} = \mu_i + \varepsilon_{ij}, \begin{cases} i = 1,2,...,a \\ j = 1,2,...,n_i \end{cases}$$

$$y_{ij} = \mu + \tau_i + \varepsilon_{ij}, \begin{cases} i = 1,2,...,a \\ j = 1,2,...,n_i \end{cases}$$

$$y_{ij} \sim N(\mu + \tau_i, \sigma^2)$$

a: number of levels of the factor,
n_i :is the number of observations on the i^{th} level of the factor,
y_{ij} :is the ij^{th} observation and

ε_{ij} :random experimental error, normally independently distributed with mean 0 and variance σ^2

μ: overall mean $= \dfrac{\sum_{i=1}^{a} \mu_i}{a}$

μ_i: treatment mean

τ_i: i^{th} treatment effect $= \overline{y_{i.}} - y_{..}$ where $\sum_{i=1}^{a} \tau_i = 0$

Test for Treatment Effect:

H_0: $\tau_1 = \tau_2 = \ldots = \tau_a = 0$

H_a: at least one τ_i is different from zero.

ANOVA:

Source of Variation	Degrees of Freedom	Sum of Squares
Treatment	t - 1	$SS_{TREATMENT} = \sum_{i=1}^{a} n_i \left(\overline{y_{i.}} - \overline{y_{..}}\right)^2 = \left(\sum_{i=1}^{a} \frac{y_{i.}^{\,2}}{n_i}\right) - \frac{y_{..}^{\,2}}{N}$
Error	n_T - 1	$SS_{ERROR} = \sum_{i=1}^{a}\sum_{j=1}^{n_i} \left(y_{ij} - \overline{y_{i.}}\right)^2 = SS_T - SS_{TREATMENT}$
Total	n_T - t	$SS_T = \sum_{i=1}^{a}\sum_{j=1}^{n_i} \left(y_{ij} - \overline{y_{..}}\right)^2 = \left(\sum_{i=1}^{a}\sum_{j=1}^{n_i} y_{ij}^{\,2}\right) - \frac{y_{..}^{\,2}}{N}$

6.3 RANDOMISED BLOCK DESIGN (RBD):

Treatment level	Observations				Total
1	y_{11}	y_{12}	y_{13}	y_{14}	$y_{1.}$
2	y_{21}	y_{22}	y_{23}	y_{24}	$y_{2.}$
3	y_{31}	y_{32}	y_{33}	y_{34}	$y_{3.}$

Treatments: = a

Factors: = 1

Replications per treatment: = b

Total Treatments: $= N = ab$

Mathematical Model:

$$y_{ij} = \mu_{ij} + \varepsilon_{ij}, \begin{cases} i = 1,2,\ldots,a \\ j = 1,2,\ldots,b \end{cases}$$

$$y_{ij} = \mu + \tau_i + \beta_j + \varepsilon_{ij}, \begin{cases} i = 1,2,\ldots,a \\ j = 1,2,\ldots,n_i \end{cases}$$

a: number of levels of the factor,

b :is the number of blocks

y_{ij} :is the ij^{th} observation and

ε_{ij} :random experimental error, normally independently distributed with mean 0 and variance σ^2

μ: overall mean

τ_i: i[th] treatment effect
β_j: j[h] block effect

Test for Treatment Effect:

H_0: $\tau_1 = \tau_2 = \ldots = \tau_a = 0$
H_a: at least one τ_i is different from zero.

Test for Block Effect:

H_0: $\beta_1 = \beta_2 = \ldots = \beta_b = 0$
H_a: at least one β_i is different from zero.

Relative Efficiency: $$RE_{RBD/CRD} = \frac{MSE_{CRD}}{MSE_{RBD}} = \frac{(b-1)MS_{BLOCK} + b(a-1)MSE}{(ba-1)MSE}$$

ANOVA:

Source of Variation	Degrees of Freedom	Sum of Squares
Treatment	a-1	$SS_{TREATMENT} = b\sum_{i=1}^{a}\left(\overline{y_{i.}} - \overline{y_{..}}\right)^2 = \frac{1}{b}\left(\sum_{i=1}^{a} y_{i.}^{\ 2}\right) - \frac{y_{..}^{\ 2}}{N}$
Block	b-1	$SS_{BLOCK} = a\sum_{i=1}^{b}\left(\overline{y_{.j}} - \overline{y_{..}}\right)^2 = \frac{1}{a}\left(\sum_{i=1}^{b} y_{.j}^{\ 2}\right) - \frac{y_{..}^{\ 2}}{N}$
Error	(a-1)(b-1)	$SS_{ERROR} = \sum_{i=1}^{a}\sum_{j=1}^{b}\left(y_{ij} - \overline{y_{i.}} - \overline{y_{.j}} + \overline{y_{..}}\right)^2 = SS_T - SS_{TREATMENT} - SS_{BLOCKS}$
Total	ab-1	$SS_T = \sum_{i=1}^{a}\sum_{j=1}^{b}\left(y_{ij} - \overline{y_{..}}\right)^2 = \left(\sum_{i=1}^{a}\sum_{j=1}^{b} y_{ij}^{\ 2}\right) - \frac{y_{..}^{\ 2}}{N}$

6.4 LATIN SQUARE DESIGN (LSD):

a=4 design	1	2	3	4
1	y_{111}	y_{221}	y_{331}	y_{441}
2	y_{212}	y_{322}	y_{432}	y_{142}
3	y_{313}	y_{423}	y_{133}	y_{243}
4	y_{414}	y_{124}	y_{234}	y_{344}

Treatments: = a
Factors: = 1
Replications per treatment: = a
Total Treatments: $= N = a^2$
Mathematical Model:

$$y_{ijk} = \mu_{ijk} + \varepsilon_{ijk}, \begin{cases} i = 1,2,\ldots,a \\ j = 1,2,\ldots,a \\ k = 1,2,\ldots,a \end{cases}$$

$$y_{ij} = \mu + \tau_i + \beta_j + \gamma_k + \varepsilon_{ij}, \begin{cases} i = 1,2,\ldots,a \\ j = 1,2,\ldots,a \\ k = 1,2,\ldots,a \end{cases}$$

a: number of levels of the factor,
y_{ijk} :is the ijk[th] observation and

ε_{ijk} :random experimental error, normally independently distributed with mean 0 and variance σ^2
μ: overall mean
τ_i: i^{th} treatment effect
β_j: j^{th} row effect
γ_k: k^{th} column effect

Test for Treatment Effect:

H_0: $\tau_1 = \tau_2 = \ldots = \tau_a = 0$
H_a: at least one τ_i is different from zero.

Relative Efficiency:

$$RE_{LSD/CRD} = \frac{MSE_{CRD}}{MSE_{LSD}} = \frac{MS_{ROWS} + MS_{COLUMNS} + (a-1)MSE}{(a+1)MSE}$$

ANOVA:

Source of Variation	Degrees of Freedom	Sum of Squares
Treatment	a-1	$SS_{TREATMENT} = a\sum_{i=1}^{a}\left(\overline{y_{i..}} - \overline{y_{...}}\right)^2$
Rows	a-1	$SS_{ROWS} = a\sum_{i=1}^{a}\left(\overline{y_{.j.}} - \overline{y_{...}}\right)^2$
Columns	a-1	$SS_{COLUMNS} = a\sum_{i=1}^{a}\left(\overline{y_{..k}} - \overline{y_{...}}\right)^2$
Error	$a^2 - 3a + 2$	$SS_{ERROR} = \sum_{i=1}^{a}\sum_{j=1}^{a}\sum_{k=1}^{a}\left(y_{ijk} - \overline{y_{i..}} - \overline{y_{.j.}} - \overline{y_{..k}} + 2\overline{y_{...}}\right)^2$
Total	$a^2 - 1$	$SS_T = \sum_{i=1}^{a}\sum_{j=1}^{a}\sum_{k=1}^{a}\left(y_{ijk} - \overline{y_{...}}\right)^2$

6.5 ANALYSIS OF COVARIANCE:

Mathematical Model:

$$y_{ij} = \mu + \tau_i + \beta\left(x_{ij} - \bar{x}\right) + \varepsilon_{ij}, \begin{cases} i = 1,2,\ldots,a \\ j = 1,2,\ldots,n_i \end{cases}$$

a: number of levels of the factor,
n_i :is the number of observations on the i^{th} level of the factor,
y_{ij} :is the ij^{th} observation and
ε_{ij} :random experimental error, normally independently distributed with mean 0 and variance σ^2
μ: overall mean
τ_i: i^{th} treatment effect
β: linear regression coefficient indicating dependency of y_{ij} on x_{ij}
x_{ij} :is the ij^{th} covariate

Assumptions:

- Treatment do not influence the covariate
  - May be obvious from the nature of the covariates.
  - Test through ANOVA on Covariates.
- The regression coefficient β is the same for all treatments
  - Perform analysis of variance on covariates
- The relationship between the response y and covariate x is linear.

- For each treatment fit a linear regression model and assess its quality

6.6 RESPONSE SURFACE METHODOLOGY:

Definition: Creating a design such that it will optimise the response

1st order:

$$y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \ldots + \beta_k x_k + \varepsilon$$

2nd order:

$$y = \beta_0 + \sum_{i=1}^{k} \beta_i x_i + \sum_{i=1}^{k} \beta_i x_i^2 + \sum_{i<j} \beta_{ij} x_i x_j + \varepsilon$$

Common Designs:

- Designs for first order model
  - 2 level factorial designs + some centre points
- Designs for second order model
  - 2 level factorial designs + some centre points + axial runs (central composite designs)
  - 3 levels designs; 2 level factorial designs and incomplete block designs (Box-Behnken designs)
- Latin Hypercube Designs
  - Space filling Multilevel designs

Criterion for determining the optimatility of a design:

X Matrix is determined as the coefficients of β in whatever order studied.

D-optimality $|(X'X)^{-1}|$

A-optimality $Trace[(X'X)^{-1}]$

G-optimality $\frac{NV[\hat{y}(x)]}{\sigma^2}$

V-optimality - Prediction variance at a set of points

6.7 FACTORIAL OF THE FORM 2^n:

General Definition: A 2^n consists of n factors each studied at 2 levels.

Contrasts for a 2^2 design:

$$M_A = \frac{ab - b + a - I}{2n}; M_B = \frac{ab - a + b - I}{2n}; M_{AB} = \frac{ab - a - b + I}{2n}$$

Sum of Squares for a 2^2 design:

$$SS_A = \frac{(ab - b + a - I)^2}{2^2 n}; SS_B = \frac{(ab - a + b - I)^2}{2^2 n}; SS_{AB} = \frac{(ab - a - b + I)^2}{2^2 n}$$

$$SS_{TREATMENT} = SS_A + SS_B + SS_{AB}$$

Hypothesis for a CRD 2^2 design:

Test for interaction effect of factor AB:

$H_0: ab_{ij} = 0 \ \forall \text{ i, j} = 1, 2$

$H_1: ab_{ij} \neq 0$ for at least one i, j

And only if the interaction is not significant:

Test for Main effect of factor A:

$H_0: a_i = 0 \ \forall \text{ i} = 1, 2$

$H_1: a_i \neq 0$ for at least one i

Test for Main effect of factor B:

$H_0: b_i = 0 \ \forall \text{ i} = 1, 2$

$H_1: b_i \neq 0$ for at least one i

Hypothesis for a RBD 2^2 design:

Test for block effect:

H_0: β_k=0 for all k

H_a: at least one $\beta_k \neq 0$

Test for interaction effect of factor AB:

H_0: ab_{ij}=0 for all (i,j)

H_a: at least one $ab_{ij} \neq 0$

Test for Main effect of factor A:

H_0: $a_1 = a_2 = 0$

H_a: at least one $a_i \neq 0$

Test for Main effect of factor B:

H_0: $b_1 = b_2 = 0$

H_a: at least one $a_i \neq 0$

6.8 GENERAL FACTORIAL:

General Definition: Factor A at x_A levels, Factor B at x_B levels, Factor C at x_C levels, etc.

Order: $x_A \times x_B \times x_C \times ...$

Degrees of freedom for Main Effects:

df(A) = x_A - 1
df(B) = x_B – 1
df(C) = x_C – 1
etc.

Degrees of freedom for Higher Order Effects:

df(AB)= df(A) df(B) etc.
df(ABC)= df(A) df(B) df(C) etc.

6.9 ANOVA ASSUMPTIONS:

Assumptions:

- Normality
  - ANOVA applied to the absolute deviation of a response from the mean
  - Plot of observed values against expected value => Linear plot will imply normality
- Constant Variance
  - Plot of residual agains predicted values => Random plot with no vertical funnelling structure will imply constant variance
  - Levene’s Test
- Independence
  - Plot of residuals in time sequence => Random plot will imply independence

Levene's Test:

$$d_{ij} = \left| y_{ij} - \psi_i \right| \begin{cases} i = 1,2,...,a \\ j = 1,2,...,n_i \end{cases}$$

Where ψ_i =

- Mean of observations at the i[th] level
- Or Median of observations at the i[th] level
- Or 10% trimmed mean of observations at the i[th] level.

6.10 CONTRASTS:

Linear Contrast: $l = \alpha_1\mu_1 + \alpha_2\mu_2 + ... + \alpha_n\mu_n ; \sum_{i=1}^{a} \alpha_i = 0$

Estimated Mean of Contrast: $\hat{E}(\hat{l}) = \sum_{i=1}^{a} \alpha_i \mu_i$

Estimated Variance of Contrast: $\hat{v}(\hat{l}) = s^2 \sum_{i=1}^{a} \frac{\alpha_i^2}{n_i}$ where s^2 is MSE of corresponding

F of Contrast: $f = \frac{\hat{E}(\hat{l})}{\hat{v}(\hat{l})} = \frac{\sum_{i=1}^{a} \alpha_i \mu_i}{s^2 \sum_{i=1}^{a} \frac{\alpha_i^2}{n_i}}$

Orthogonal Contrasts: if $\sum_i \alpha_i \beta_i = 0$

6.11 POST ANOVA MULTIPLE COMPARISONS:

When the null hypothesis is rejected and the optimum treatment combination needs to be found,

Bonderroni Method:

- Let m be the number of different treatments performed
- Perform each individual test at level of significance $\frac{\alpha}{m}$
- Difference of the contrast is significant if F(l) > Critical Value of F using new level of significance

Fisher's Least Significant Difference:

- $LSD_{ij} = t_{\alpha/2, df_{TREATMENT}, df_{ERROR}} s \sqrt{\frac{1}{n_i} + \frac{1}{n_j}} = t_{\alpha/2, df_{TREATMENT}, df_{ERROR}} \sqrt{MSE} \sqrt{\frac{1}{n_i} + \frac{1}{n_j}}$
- Difference is significant if $\left| \overline{y_{i.}} - \overline{y_{j.}} \right| > LSD$

Tukey's W Procedure:

- Difference is significant if $\left| \overline{y_{i.}} - \overline{y_{j.}} \right| \geq q_{\alpha}(t, v) \sqrt{\frac{MSE}{n}}$
- q is a conservative statistic read from the tables

Scheffe's Method:

- $S = \sqrt{\hat{v}(\hat{l})} \sqrt{(a-1) F_{\alpha, df_1, df_2}}$
- a is the number of treatments, df_1 = a-1, df_2 = degrees of freedom for MSE

- Difference is significant if $\left|\hat{E}(\hat{l})\right| > S$

PART 7: PI

7.1 AREA:

Circle: $A = \pi r^2 = \frac{\pi d^2}{4} = \frac{Cd}{4}$

Cyclic Quadrilateral: $\sqrt{(s-a)(s-b)(s-c)(s-d)}$

Area of a sector (degrees) $A = \frac{Q}{360} \times \pi r^2$

Area of a sector (radians) $A = \frac{1}{2} r^2 \theta$

Area of a segment (degrees) $A = \frac{r^2}{2}\left(\frac{Q}{180} \times \pi - \sin Q\right)$

Area of an annulus: $A = \pi\left(r_2^{\ 2} - r_1^{\ 2}\right) = \pi\left(\frac{w}{2}\right)^2$

Ellipse: $A = \frac{\pi}{4} lw = \pi r_1 r_2$

7.2 VOLUME:

Cylinder: $V = \pi r^2 h$

Sphere: $V = \frac{4}{3} \pi r^3$

Cap of a Sphere: $V = \frac{1}{6} \pi h\left(3r_1^{\ 2} + h^2\right)$

Cone: $V = \frac{1}{3} \pi r^2 h$

Ice-cream & Cone: $V = \frac{1}{3} \pi r^2 (h + 2r)$

Doughnut: $V = 2\pi^2 r_2 r_1^{\ 2} = \frac{\pi^2}{4}(b+a)(b-a)^2$

Sausage: $V = \frac{\pi w^2}{4}\left(l - \frac{w}{3}\right)$

Ellipsoid: $V = \frac{4}{3} \pi r_1 r_2 r_3$

7.3 SURFACE AREA:

Sphere: $SA = 4\pi r^2$

Hemisphere: $SA = 3\pi r^2$

Doughnut: $SA = 4\pi^2 r_2 r_1 = \pi^2\left(b^2 - a^2\right)$

Sausage: $SA = \pi w l$

Cone: $SA = \pi r\left(r + \sqrt{r^2 + h^2}\right)$

7.4 MISELANIOUS:

Length of arc (degrees) $l = \frac{Q}{360} \times C = \frac{Q}{180} \times \pi r$

Length of chord (degrees) $l = 2r \times \sin\left(\frac{Q}{2}\right) = 2\sqrt{r^2 - h^2}$

Perimeter of an ellipse $P \approx \pi(r_1 + r_2)\left(\frac{1 + \frac{3(r_1 - r_2)^2}{(r_1 + r_2)^2}}{10 + \sqrt{4 - \frac{3(r_1 - r_2)^2}{(r_1 + r_2)^2}}}\right)$

7.6 PI:

$$\pi \approx 3.14159265358979323846264338327950288...$$

$$\pi = \frac{C}{d}$$

Archimedes' Bounds: $2^k n \sin\left(\frac{\theta}{2^k}\right) < \pi < 2^k n \sin\left(\frac{\theta}{2^k}\right)$

John Wallis: $\frac{\pi}{2} = \frac{2}{1} \times \frac{2}{3} \times \frac{4}{3} \times \frac{4}{5} \times \frac{6}{5} \times \frac{6}{7} \times \frac{8}{7} \times \frac{8}{9} \times ... = \prod_{n=1}^{\infty} \frac{4n^2}{4n^2 - 1}$

Isaac Newton: $\frac{\pi}{6} = \frac{1}{2} + \frac{1}{2}\left(\frac{1}{3 \times 2^3}\right) + \frac{1 \times 3}{2 \times 4}\left(\frac{1}{5 \times 2^5}\right) + \frac{1 \times 3 \times 6}{2 \times 4 \times 6}\left(\frac{1}{7 \times 2^7}\right) + ...$

James Gregory: $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \frac{1}{13} - \frac{1}{15} ...$

Schulz von Strassnitzky: $\frac{\pi}{4} = \arctan\left(\frac{1}{2}\right) + \arctan\left(\frac{1}{5}\right) + \arctan\left(\frac{1}{8}\right)$

John Machin: $\frac{\pi}{4} = 4\arctan\left(\frac{1}{5}\right) - \arctan\left(\frac{1}{239}\right)$

Leonard Euler: $\frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + ...$

$$\frac{\pi}{4} = \frac{3}{4} \times \frac{5}{4} \times \frac{7}{8} \times \frac{11}{12} \times \frac{13}{12} \times \frac{17}{16} \times \frac{19}{20} \times \frac{23}{24} \times \frac{29}{28} \times \frac{31}{32} \times ...$$

where the numerators are the odd primes; each denominator is the multiple of four nearest to the numerator.

$$`\pi = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} - \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} - \frac{1}{10} + \frac{1}{11} + \frac{1}{12} - \frac{1}{13} + ...$$

If the denominator is a prime of the form 4m - 1, the sign is positive; if the denominator is 2 or a prime of the form 4m + 1, the sign is negative; for composite numbers, the sign is equal the product of the signs of its factors.

Jozef Hoene-Wronski: $\pi = \lim_{n \to \infty} \frac{4n\left((1+i)^{\left(\frac{1}{n}\right)} - (1-i)^{\left(\frac{1}{n}\right)}\right)}{i}$

Franciscus Vieta: $\frac{2}{\pi} = \frac{\sqrt{2}}{2} \times \frac{\sqrt{2+\sqrt{2}}}{2} \times \frac{\sqrt{2+\sqrt{2+\sqrt{2}}}}{2} \times ...$

Integrals:

$$\int_{-\infty}^{\infty} \operatorname{sech}(x)dx = \pi$$

$$\int_{-1}^{1} \sqrt{1-x^2}\,dx = \frac{\pi}{2}$$

$$\int_{-1}^{1} \frac{dx}{\sqrt{1-x^2}} = \pi$$

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \pi$$

$$\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}$$

$$\int_{-\infty}^{\infty} \frac{\sin(x)}{x}\,dx = \pi$$

$$\int_{0}^{1} \frac{x^4(1-x)^4}{1+x^2}\,dx = \frac{22}{7} - \pi$$

Infinite Series:

$$\sum_{k=0}^{\infty} \frac{k!}{(2k+1)!!} = \sum_{k=0}^{\infty} \frac{2^k k!^2}{(2k+1)!} = \frac{\pi}{2}$$

$$12\sum_{k=0}^{\infty} \frac{(-1)^k (6k)!(13591409+545140134k)}{(3k)!(k!)^3 640320^{3k+3/2}} = \frac{1}{\pi}$$

$$\frac{2\sqrt{2}}{9801}\sum_{k=0}^{\infty} \frac{(4k)!(1103+26390k)}{(k!)^4 396^{4k}} = \frac{1}{\pi}$$

$$\frac{\sqrt{3}}{6^5}\sum_{k=0}^{\infty} \frac{((4k)!)^2(6k)!}{9^{k+1}(12k)!(2k)!}\left(\frac{127169}{12k+1} - \frac{1070}{12k+5} - \frac{131}{12k+7} + \frac{2}{12k+11}\right) = \pi$$

$$\sum_{k=0}^{\infty} \frac{1}{16^k}\left(\frac{4}{8k+1} - \frac{2}{8k+4} - \frac{1}{8k+5} - \frac{1}{8k+6}\right) = \pi$$

$$\frac{1}{2^6}\sum_{n=0}^{\infty} \frac{(-1)^n}{2^{10n}}\left(-\frac{2^5}{4n+1} - \frac{1}{4n+3} + \frac{2^8}{10n+1} - \frac{2^6}{10n+3} - \frac{2^2}{10n+5} - \frac{2^2}{10n+7} + \frac{1}{10n+9}\right) = \pi$$

See also: Zeta Function within Part 17

Continued Fractions:

$$\pi = 3 + \cfrac{1^2}{6 + \cfrac{3^2}{6 + \cfrac{5^2}{6 + \cfrac{7^2}{6 + \ddots}}}}$$

$$\pi = \cfrac{4}{1 + \cfrac{1^2}{3 + \cfrac{2^2}{5 + \cfrac{3^2}{7 + \cfrac{4^2}{9 + \ddots}}}}}$$

$$\pi = \cfrac{4}{1 + \cfrac{1^2}{2 + \cfrac{3^2}{2 + \cfrac{5^2}{2 + \cfrac{7^2}{2 + \ddots}}}}}$$

7.7 CIRCLE GEOMETRY:

Radius of Circumscribed Circle for Rectangles: $r = \dfrac{\sqrt{a^2 + b^2}}{2}$

Radius of Circumscribed Circle for Squares: $r = \dfrac{a}{\sqrt{2}}$

Radius of Circumscribed Circle for Triangles: $r = \dfrac{a}{2\sin A}$

Radius of Circumscribed Circle for Quadrilaterals:

$$r = \frac{1}{4} \times \sqrt{\frac{(ab+cd)(ac+bd)(ad+bc)}{(s-a)(s-b)(s-c)(s-d)}}$$

Radius of Inscribed Circle for Squares: $r = \dfrac{a}{2}$

Radius of Inscribed Circle for Triangles: $r = \dfrac{A}{s}$

Radius of Circumscribed Circle: $r = \dfrac{a}{2\sin\left(\dfrac{180}{n}\right)}$

Radius of Inscribed Circle: $r = \dfrac{a}{2\tan\left(\dfrac{180}{n}\right)}$

7.8 ABBREVIATIONS (7.1, 7.2, 7.3, 7.4, 7.5, 7.6, 7.7):

A=Angle ‘A’
A=Area
a=side ‘a’
B=Angle ‘B’
b=side ‘b’
B=Angle ‘B’
c=side ‘c’
C=circumference
d=diameter
d=side ‘d’
h=shortest length from the center to the chord
r=radius
r_1=radius 1 ($r_1 < r_2$)
r_2=radius 2 ($r_2 < r_3$)
r_3=radius 3
l=length
n=number of sides
P=perimeter
Q=central angle
s=semi-perimeter
w=width
w=length of chord from r_1

7.9 CRESCENT GEOMETRY:

Area of a lunar crescent: $A = \dfrac{1}{4}\pi cd$

Area of an eclipse crescent:

$$A = w^2\left(\pi - \frac{2\pi\left(\cos^{-1}\left(\dfrac{w^2+l^2-b^2}{2wl}\right)\right)}{360} + \frac{\sin 2\left(\cos^{-1}\left(\dfrac{w^2+l^2-b^2}{2wl}\right)\right)}{2}\right)$$

$$-b^2\left(\pi - \frac{2\pi\left(\cos^{-1}\left(\dfrac{w^2+l^2-b^2}{2wl}\right)\right)}{360} + \frac{\sin 2\left(\cos^{-1}\left(\dfrac{w^2+l^2-b^2}{2wl}\right)\right)}{2}\right)$$

7.10 ABBREVIATIONS (7.9):

A=Area
b=radius of black circle
c=width of the crescent
d=diameter
l=distance between the centres of the circles
w=radius of white circle

PART 8: APPLIED FIELDS:

8.1 MOVEMENT:

Stopping distance: $s = \frac{v^2}{-2a}$

Centripetal acceleration: $a = \frac{v^2}{r}$

Centripetal force: $F_C = ma = \frac{mv^2}{r}$

Dropping time : $t = \sqrt{\frac{2h}{g}}$

Force: $F = \frac{ma}{\left(1 - \frac{v^2}{c^2}\right)^{\frac{3}{2}}}$

Kinetic Energy: $E_k = \frac{1}{2}mv^2$

Maximum height of a cannon: $h = \frac{(u \sin\theta)^2}{g}$

Pendulum swing time: $t = 2\pi\sqrt{\frac{l}{g}}$

Potential Energy: $E_p = mgh$

Range of a cannon: $s = t(u\cos\theta) = \frac{2u\sin\theta}{g} \times (u\cos\theta)$

Time in flight of a cannon: $t = \frac{2u\sin\theta}{g}$

Universal Gravitation: $F = G\frac{m_1 m_2}{r^2}$

ABBREVIATIONS (8.1):

a=acceleration (negative if retarding)
c=speed of light (3×10^8 ms^{-1})
E_k=Kinetic Energy
E_p=potential energy
F=force
g=gravitational acceleration (≈9.81 on Earth)
G=gravitational constant = 6.67×10^{-11}
h=height
l=length of a pendulum
m=mass
m_1=mass 1
m_2=mass 2

r=radius
r=distance between two points
s=distance
t=time
u=initial speed
v=final speed
θ=the angle

8.2 CLASSICAL MECHANICS:

Newton's Laws:

First law: If an object experiences no net force, then its velocity is constant; the object is either at rest (if its velocity is zero), or it moves in a straight line with constant speed (if its velocity is nonzero).

$$\sum \mathbf{F} = 0 \Rightarrow \frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = 0.$$

Second law: The acceleration a of a body is parallel and directly proportional to the net force F acting on the body, is in the direction of the net force, and is inversely proportional to the mass m of the body, i.e., F = ma.

$$\mathbf{F} = \frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} = \frac{\mathrm{d}(m\mathbf{v})}{\mathrm{d}t}.$$

$$\mathbf{J} = \int_{\Delta t} \mathbf{F}\,\mathrm{d}t.$$

$$\mathbf{J} = \Delta\mathbf{p} = m\Delta\mathbf{v}.$$

$$\mathbf{F} + \mathbf{u}\frac{\mathrm{d}m}{\mathrm{d}t} = m\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t}$$

$$\mathbf{F} = m\mathbf{a}.$$

Third law: When two bodies interact by exerting force on each other, these forces (termed the action and the reaction) are equal in magnitude, but opposite in direction.

$$\sum \mathbf{F}_{a,b} = -\sum \mathbf{F}_{b,a}$$

Inertia:

$$p = mv$$

$$p = p_1 + p_2$$
$$= m_1v_1 + m_2v_2\,.$$

$$\Delta p = F\Delta t\,.$$

$$\Delta p = \int_{t_1}^{t_2} F(t)dt\,.$$

$$\frac{dp_1}{dt} = -\frac{dp_2}{dt},$$

$$\frac{d}{dt}(p_1 + p_2) = 0.$$

$$m_1u_1 + m_2u_2 = m_1v_1 + m_2v_2$$

$$\tfrac{1}{2}m_1u_1^2 + \tfrac{1}{2}m_2u_2^2 = \tfrac{1}{2}m_1v_1^2 + \tfrac{1}{2}m_2v_2^2\,.$$

$$v_1 = \left(\frac{m_1 - m_2}{m_1 + m_2}\right) u_1 + \left(\frac{2m_2}{m_1 + m_2}\right) u_2$$

$$v_2 = \left(\frac{m_2 - m_1}{m_1 + m_2}\right) u_2 + \left(\frac{2m_1}{m_1 + m_2}\right) u_1 .$$

$$p_x = mv_x$$

$$p_y = mv_y$$

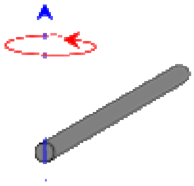
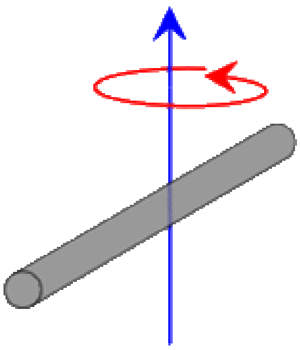
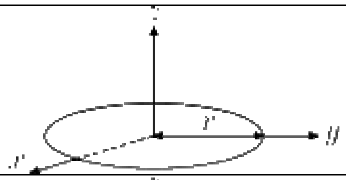
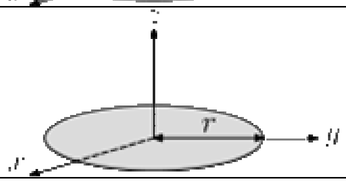
$$p_z = mv_z.$$

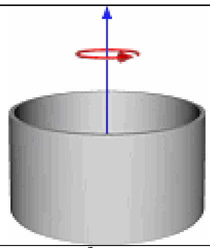
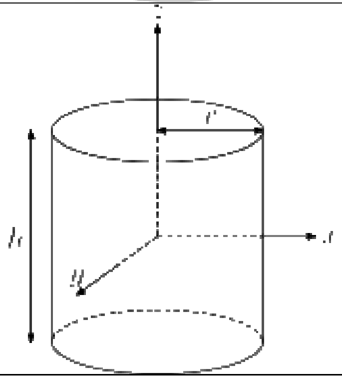
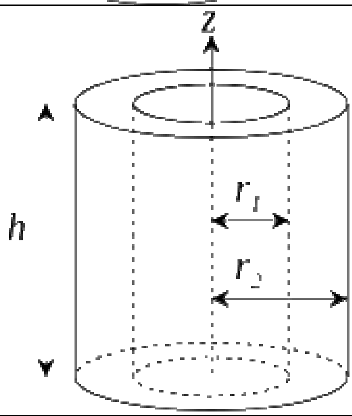
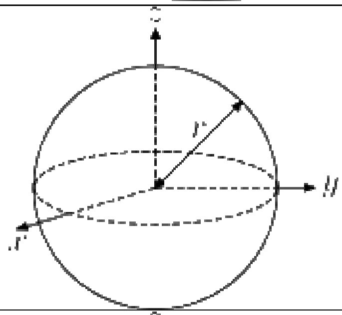
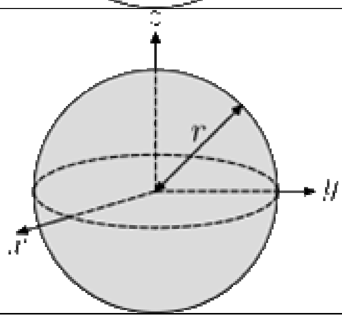
Moments of Inertia:

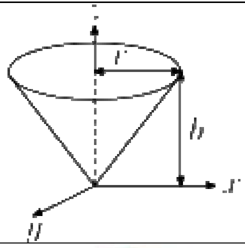
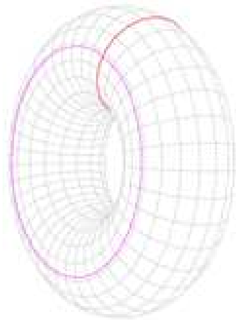
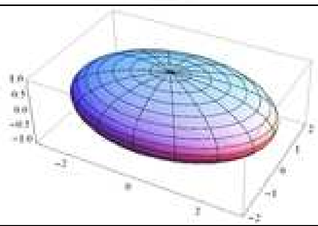
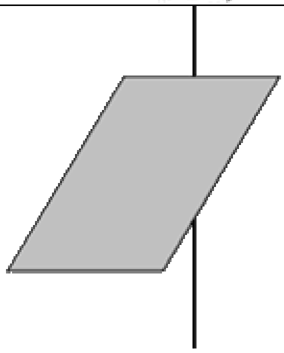
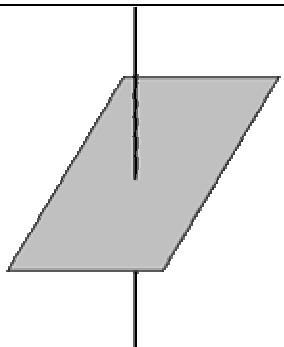
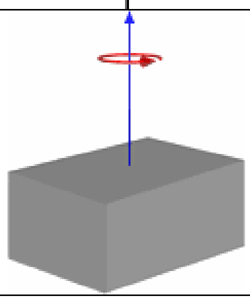
$$I_{\text{net}} = \sum_j I_j$$

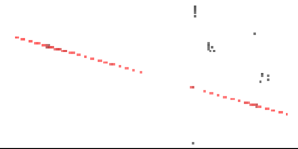
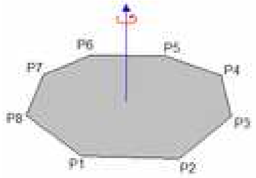
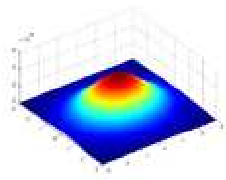
$$I_d = I_{\text{com}} + Md^2$$

$$I_k \leq I_i + I_j$$

Description	Diagram	Formulae
Two point masses, *M* and *m*, with reduced mass μ and separated by a distance, *x*.		$I = mr^2$
Rod of length *L* and mass *m* (Axis of rotation at the end of the rod)		$I = \frac{Mm}{M+m}x^2 = \mu x^2$
Rod of length *L* and mass *m*		$I_{\text{end}} = \frac{mL^2}{3}$
Thin circular hoop of radius *r* and mass *m*		$I_{\text{center}} = \frac{mL^2}{12}$
Thin circular hoop of radius *r* and mass *m*		$I_z = mr^2$ $I_x = I_y = \frac{mr^2}{2}$
Thin, solid disk of radius *r* and mass *m*		$I_z = \frac{mr^2}{2}$ $I_x = I_y = \frac{mr^2}{4}$

Thin cylindrical shell with open ends, of radius *r* and mass *m*		$I = mr^2$
Solid cylinder of radius *r*, height *h* and mass *m*		$I_z = \frac{mr^2}{2}$ $I_x = I_y = \frac{1}{12}m\left(3r^2 + h^2\right)$
Thick-walled cylindrical tube with open ends, of inner radius r_1, outer radius r_2, length *h* and mass *m*		$I_z = \frac{1}{2}m\left({r_1}^2 + {r_2}^2\right)$ $I_x = I_y = \frac{1}{12}m\left[3\left({r_2}^2 + {r_1}^2\right) + h^2\right]$ or when defining the normalized thickness $t_n = t/r$ and letting $r = r_2$, then $I_z = mr^2\left(1 - t_n + \frac{1}{2}{t_n}^2\right)$
Sphere (hollow) of radius *r* and mass *m*		$I = \frac{2mr^2}{3}$
Ball (solid) of radius *r* and mass *m*		$I = \frac{2mr^2}{5}$

Right circular cone with radius *r*, height *h* and mass *m*		$I_z = \frac{3}{10}mr^2$ $I_x = I_y = \frac{3}{5}m\left(\frac{r^2}{4} + h^2\right)$
Torus of tube radius *a*, cross-sectional radius *b* and mass *m*.		About a diameter: $\frac{1}{8}\left(4a^2 + 5b^2\right)m$ About the vertical axis: $\left(a^2 + \frac{3}{4}b^2\right)m$
Ellipsoid (solid) of semiaxes *a*, *b*, and *c* with axis of rotation *a* and mass *m*		$I_a = \frac{m(b^2 + c^2)}{5}$
Thin rectangular plate of height *h* and of width *w* and mass *m* (Axis of rotation at the end of the plate)		$I_e = \frac{mh^2}{3} + \frac{mw^2}{12}$
Thin rectangular plate of height *h* and of width *w* and mass *m*		$I_c = \frac{m(h^2 + w^2)}{12}$
Solid cuboid of height *h*, width *w*, and depth *d*, and mass *m*		$I_h = \frac{1}{12}m\left(w^2 + d^2\right)$ $I_w = \frac{1}{12}m\left(h^2 + d^2\right)$ $I_d = \frac{1}{12}m\left(h^2 + w^2\right)$

Solid cuboid of height *D*, width *W*, and length *L*, and mass *m* with the longest diagonal as the axis.		$I = \frac{m\left(W^2D^2 + L^2D^2 + L^2W^2\right)}{6\left(L^2 + W^2 + D^2\right)}$
Plane polygon with vertices $\vec{P}_1$, $\vec{P}_2$, $\vec{P}_3$, ..., $\vec{P}_N$ and mass m uniformly distributed on its interior, rotating about an axis perpendicular to the plane and passing through the origin.		
	$I = \frac{m}{6} \frac{\sum\limits_{n=1}^{N-1} \lVert\vec{P}_{n+1} \times \vec{P}_n\rVert\left(\left(\vec{P}_{n+1} \cdot \vec{P}_{n+1}\right) + \left(\vec{P}_{n+1} \cdot \vec{P}_n\right) + \left(\vec{P}_n \cdot \vec{P}_n\right)\right)}{\sum\limits_{n=1}^{N-1} \lVert\vec{P}_{n+1} \times \vec{P}_n\rVert}$	
Infinite disk with mass normally distributed on two axes around the axis of rotation (i.e. $\rho(x,y) = \frac{m}{2\pi ab} e$ Where : $\rho(x,y)$ is the mass-density as a function of x and y).		$I = m(a^2 + b^2)$

Velocity and Speed:

$$v_{AVE} = \frac{\Delta P}{\Delta t}$$

$$\mathbf{V} = \lim_{\Delta t \to 0} \frac{\Delta \mathbf{P}}{\Delta t} = \frac{d\mathbf{P}}{dt} = \dot{\mathbf{P}} = \dot{x}_p\vec{i} + \dot{y}_P\vec{j} + \dot{z}_P\vec{k}.$$

$$|\mathbf{V}| = |\dot{\mathbf{P}}| = \frac{ds}{dt},$$

$$\mathbf{V}(t) = \int_0^t \mathbf{A}dt = \mathbf{A}t + \mathbf{V}_0.$$

$$v^2 = v_x^2 + v_y^2 + v_z^2.$$

Acceleration:

$$a_{AVE} = \frac{\Delta V}{\Delta t}$$

$$\mathbf{A} = \lim_{\Delta t \to 0} \frac{\Delta \mathbf{V}}{\Delta t} = \frac{d\mathbf{V}}{dt} = \dot{\mathbf{V}} = \ddot{\mathbf{P}} = \ddot{x}_p\vec{i} + \ddot{y}_P\vec{j} + \ddot{z}_P\vec{k}.$$

Trajectory (Displacement):

$$\mathbf{P}(t) = \int_0^t \mathbf{V}(t)dt = \int (\mathbf{A}t + \mathbf{V}_0)dt = \tfrac{1}{2}\mathbf{A}t^2 + \mathbf{V}_0 t + \mathbf{P}_0.$$

$$\mathbf{P}(t) = \mathbf{P}_0 + \left(\frac{\mathbf{V} + \mathbf{V}_0}{2}\right) t.$$
$$(\mathbf{P} - \mathbf{P}_0) \cdot \mathbf{A}t = (\mathbf{V} - \mathbf{V}_0) \cdot \frac{\mathbf{V} + \mathbf{V}_0}{2} t \ ;$$

Kinetic Energy:

$$E_k = \tfrac{1}{2} m v^2$$
$$E_k = \frac{p^2}{2m}$$
$$E_k = \int \mathbf{F} \cdot d\mathbf{x} = \int \mathbf{v} \cdot d(m\mathbf{v}) = \int d\left(\frac{mv^2}{2}\right) = \frac{mv^2}{2}.$$
$$E_k = E_i + \frac{MV^2}{2}.$$
$$E_r = \int \frac{v^2 dm}{2} = \int \frac{(r\omega)^2 dm}{2} = \frac{\omega^2}{2} \int r^2 dm = \frac{\omega^2}{2} I = \tfrac{1}{2} I \omega^2$$

Centripetal Force:

$$F = ma_c = \frac{mv^2}{r}$$
$$F = mr\omega^2 .$$
$$F = mr\frac{4\pi^2}{T^2} .$$

Circular Motion:

$$\omega = \frac{2\pi}{T} = 2\pi f = \frac{|v|}{|r|} ;$$
$$\omega_{cyc} = \omega_{rad} / 2\pi$$
$$\omega_{cyc} = \omega_{deg} / 360$$
$$v = \frac{2\pi r}{T} = \omega r$$
$$\theta = 2\pi \frac{t}{T} = \omega t$$
$$a = \frac{v^2}{r}$$
$$\alpha = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2}$$, or
$$\alpha = \frac{a_T}{r} ,$$
$$\alpha = \frac{\tau}{I} .$$

Angular Momentum:

$$\mathbf{L} = I\boldsymbol{\omega} .$$
$$\mathbf{L} = \mathbf{r} \times m\mathbf{v} .$$
$$\mathbf{L} = \mathbf{r} \times \mathbf{p}$$

$$\mathbf{L} = \sum_i \mathbf{R}_i \times m_i \mathbf{V}_i$$

$$\mathbf{L} = \sum_i \mathbf{R}_i \times m_i \mathbf{V}_i$$

Torque:

$$\boldsymbol{\tau} = \frac{d\mathbf{L}}{dt} = \frac{d\mathbf{r}}{dt} \times \mathbf{p} + \mathbf{r} \times \frac{d\mathbf{p}}{dt} = 0 + \mathbf{r} \times \mathbf{F} = \mathbf{r} \times \mathbf{F}$$

$$\tau = rF \sin\theta$$

$$\boldsymbol{\tau}_{\mathrm{net}} = \frac{d\mathbf{L}}{dt} = \frac{d(I\boldsymbol{\omega})}{dt} = I\frac{d\boldsymbol{\omega}}{dt} = I\boldsymbol{\alpha},$$

$$|\tau| = (\text{moment arm})(\text{force}).$$

Work:

$$W = \int_{\theta_1}^{\theta_2} \tau \; d\theta,$$

Laws of Conservation:

Momentum: $\frac{d}{dt}(p_1 + p_2) = 0.$

Energy: $\sum E_{IN} = \sum E_{OUT}$

Force: $\sum F_{NET} = 0 \Rightarrow \sum F_{UP} = \sum F_{DN}, \sum F_L = \sum F_R, \sum cm = \sum acm$

ABBREVIATIONS (8.2)

a=acceleration
E_K=Kinetic Energy
E_r=rotational kinetic energy
F=force
I=mass moment of inertia
J=impulse
L=angular momentum
m=mass
P=path
p=momentum
t=time
v=velocity
W=work
τ=torque

8.3 RELATIVISTIC EQUATIONS:

Kinetic Energy:

$$E_k = \frac{mc^2}{\sqrt{1-(v/c)^2}} - mc^2$$

Momentum:

$$p = \frac{mv}{\sqrt{1-(v/c)^2}}$$

Time Dilation:

$$\Delta t' = \frac{\Delta t}{\sqrt{1-v^2/c^2}}$$

Length Contraction:

$$L = L_0 \cdot \sqrt{1-\frac{v^2}{c^2}}.$$

Relativistic Mass:

$$m = \frac{m_0 c^2}{\sqrt{1-\frac{v^2}{c^2}}}$$

.

8.4 ACCOUNTING:

Profit: $p = s - c$

Profit margin: $m = \frac{p}{c} \times 100$

Simple Interest: $= P(1+tr)$

Compound Interest: $= P(1+r)^t$

Continuous Interest: $= Pe^{rt}$

ABBREVIATIONS (8.4):

c=cost
I=interest
m=profit margin (%)
p=profit
P=premium
r=rate
s=sale price
t=time

8.5 MACROECONOMICS:

GDP: y = AE = AD = C + I + G + NX
y = Summation of all product quantities multiplied by cost

RGDP: RGDP = Summation of all product quantities multiplied by base year cost

NGDP: NGDP = Summation of all product quantities multiplied by current year cost

Growth: $\text{Growth} = \frac{RGDP_{CURRENT} - RGDP_{BASE}}{RGDP_{BASE}} \times 100$

Net Exports: NX = X - M

Working Age Population: WAP = Labor Force + Not in Labor Force

Labor Force: LF = Employed + Unemployed

Unemployment: UE = Frictional + Structural + Cyclical

Natural Unemployment: NUE = Frictional + Structural

Unemployment Rate: $\Delta UE\% = \frac{UE}{LF} \times 100$

Employment Rate: $\Delta E\% = \frac{E}{LF} \times 100$

Participation Rate: $\Delta P\% = \frac{LF}{WAP} \times 100 = \frac{UE + E}{WAP} \times 100$

CPI: CPI = Indexed Average Price of all Goods and Services

Inflation Rate: Inflation Rate = $\frac{CPI_{CURRENT} - CPI_{BASE}}{CPI_{BASE}} \times 100$

ABBREVIATIONS (8.5)

AD=Aggregate Demand
AE=Aggregate Expenditure
C=Consumption
CPI=Consumer Price Index
E=Employed
G=Government
I=Investment
LF=Labor Force
M=Imports
NGDP=Nominal GDP
NUE=Natural Unemployment
NX=Net Export
P=Participation
RGDP=Real GDP (Price is adjusted to base year)
UE=Unemployed
WAP=Working Age Population
X=Exports
Y=GDP

PART 9: TRIGONOMETRY

9.1 CONVERSIONS:

Degrees	30°	60°	120°	150°	210°	240°	300°	330°
Radians	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	$\frac{7\pi}{6}$	$\frac{4\pi}{3}$	$\frac{5\pi}{3}$	$\frac{11\pi}{6}$
Grads	33⅓ grad	66⅔ grad	133⅓ grad	166⅔ grad	233⅓ grad	266⅔ grad	333⅓ grad	366⅔ grad
Degrees	45°	90°	135°	180°	225°	270°	315°	360°
Radians	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π	$\frac{5\pi}{4}$	$\frac{3\pi}{2}$	$\frac{7\pi}{4}$	2π
Grads	50 grad	100 grad	150 grad	200 grad	250 grad	300 grad	350 grad	400 grad

9.2 BASIC RULES:

$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$

Sin Rule: $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$ or $\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$

Cos Rule: $\cos A = \frac{b^2 + c^2 - a^2}{2bc}$ or $a^2 = b^2 + c^2 - 2bc\cos A$

Tan Rule:

$$\frac{\tan\frac{A-B}{2}}{\tan\frac{A+B}{2}} = \frac{a-b}{a+b}$$

$$\frac{\tan\frac{A-C}{2}}{\tan\frac{A+C}{2}} = \frac{a-c}{a+c}$$

$$\frac{\tan\frac{B-C}{2}}{\tan\frac{B+C}{2}} = \frac{b-c}{b+c}$$

Auxiliary Angle:

$$a\sin x \pm b\cos x = R\sin(x \pm \alpha), \quad 0 < \alpha < \frac{\pi}{2}$$

$$\text{where } R^2 = a^2 + b^2, \quad \tan\alpha = \frac{b}{a}$$

Pythagoras Theorem: $a^2 + b^2 = c^2$

Periodicy:

$$\sin(n\pi) = 0, n \in \mathrm{Z}$$

$$\cos(n\pi) = (-1)^n, n \in \mathrm{Z}$$

9.3 RECIPROCAL FUNCTIONS

$$\sec\theta = \frac{1}{\cos\theta}$$

$$\csc\theta = \frac{1}{\sin\theta}$$

$$\cot\theta = \frac{1}{\tan\theta} = \frac{\cos\theta}{\sin\theta}$$

9.4 BASIC IDENTITES:

Pythagorean Identity: $\cos^2\theta + \sin^2\theta = 1$

$$\sin(-\theta) = -\sin\theta$$
$$\cos(-\theta) = +\cos\theta$$
$$\tan(-\theta) = -\tan\theta$$
$$\csc(-\theta) = -\csc\theta$$
$$\sec(-\theta) = +\sec\theta$$
$$\cot(-\theta) = -\cot\theta$$
$$\sin(\tfrac{\pi}{2} - \theta) = +\cos\theta$$
$$\cos(\tfrac{\pi}{2} - \theta) = +\sin\theta$$
$$\tan(\tfrac{\pi}{2} - \theta) = +\cot\theta$$
$$\csc(\tfrac{\pi}{2} - \theta) = +\sec\theta$$
$$\sec(\tfrac{\pi}{2} - \theta) = +\csc\theta$$
$$\cot(\tfrac{\pi}{2} - \theta) = +\tan\theta$$
$$\sin(\pi - \theta) = +\sin\theta$$
$$\cos(\pi - \theta) = -\cos\theta$$
$$\tan(\pi - \theta) = -\tan\theta$$
$$\csc(\pi - \theta) = +\csc\theta$$
$$\sec(\pi - \theta) = -\sec\theta$$
$$\cot(\pi - \theta) = -\cot\theta$$

9.5 IDENTITIES BETWEEN RELATIONSHIPS:

$$\sin(\theta) = \pm\sqrt{1-\cos^2(\theta)} = \pm\frac{\tan(\theta)}{\sqrt{1+\tan^2(\theta)}} = \frac{1}{\csc(\theta)} = \pm\frac{\sqrt{\sec^2(\theta)-1}}{\sec(\theta)} = \pm\frac{1}{\sqrt{1+\cot^2(\theta)}}$$

$$\cos(\theta) = \pm\sqrt{1-\sin^2(\theta)} = \pm\frac{1}{\sqrt{1+\tan^2(\theta)}} = \pm\frac{\sqrt{\csc^2(\theta)-1}}{\csc(\theta)} = \frac{1}{\sec(\theta)} = \pm\frac{\cot(\theta)}{\sqrt{1+\cot^2(\theta)}}$$

$$\tan(\theta)=\pm\frac{\sin(\theta)}{\sqrt{1-\sin^2(\theta)}}=\pm\frac{\sqrt{1-\cos^2(\theta)}}{\cos(\theta)}=\pm\frac{1}{\sqrt{\csc^2(\theta)-1}}=\pm\sqrt{\sec^2(\theta)-1}=\frac{1}{\cot(\theta)}$$

$$\csc(\theta)=\frac{1}{\sin(\theta)}=\pm\frac{1}{\sqrt{1-\cos^2(\theta)}}=\pm\frac{\sqrt{1+\tan^2(\theta)}}{\tan(\theta)}=\pm\frac{\sec(\theta)}{\sqrt{\sec^2(\theta)-1}}=\pm\sqrt{1+\cot^2(\theta)}$$

$$\cot(\theta)=\pm\frac{\sqrt{1-\sin^2(\theta)}}{\sin(\theta)}=\pm\frac{\cos(\theta)}{\sqrt{1-\cos^2(\theta)}}=\frac{1}{\tan(\theta)}=\pm\sqrt{\csc^2(\theta)-1}=\pm\frac{1}{\sqrt{\sec^2(\theta)-1}}$$

$$\arctan\left(\frac{1}{x}\right)=\arctan\left(\frac{1}{x+k}\right)+\arctan\left(\frac{k}{x^2+kx+1}\right)$$

9.6 ADDITION FORMULAE:

Sine: $\sin(\alpha\pm\beta)=\sin\alpha\cos\beta\pm\cos\alpha\sin\beta$

Cosine: $\cos(\alpha\pm\beta)=\cos\alpha\cos\beta\mp\sin\alpha\sin\beta$

Tangent: $\tan(\alpha\pm\beta)=\dfrac{\tan\alpha\pm\tan\beta}{1\mp\tan\alpha\tan\beta}$

Arcsine: $\arcsin\alpha\pm\arcsin\beta=\arcsin(\alpha\sqrt{1-\beta^2}\pm\beta\sqrt{1-\alpha^2})$

Arccosine: $\arccos\alpha\pm\arccos\beta=\arccos(\alpha\beta\mp\sqrt{(1-\alpha^2)(1-\beta^2)})$

Arctangent: $\arctan\alpha\pm\arctan\beta=\arctan\left(\dfrac{\alpha\pm\beta}{1\mp\alpha\beta}\right)$

9.7 DOUBLE ANGLE FORMULAE:

Sine:

$$\sin 2\theta=2\sin\theta\cos\theta=\frac{2\tan\theta}{1+\tan^2\theta}$$

Generally,

$$\sin(nx)=\sum_{k=0}^{n}\binom{n}{k}\cos^k(x)\sin^{n-k}(x)\sin\left(\frac{1}{2}(n-k)\pi\right)$$

Cosine:

$$\cos 2\theta=\cos^2\theta-\sin^2\theta=2\cos^2\theta-1=1-2\sin^2\theta=\frac{1-\tan^2\theta}{1+\tan^2\theta}$$

Generally,

$$\cos(nx)=\sum_{k=0}^{n}\binom{n}{k}\cos^k(x)\sin^{n-k}(x)\cos\left(\frac{1}{2}(n-k)\pi\right)$$

Tangent:

$$\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta}$$

Generally.

$$\tan(nx) = \frac{\sin(nx)}{\cos(nx)} = \frac{\sum_{k=0}^{n}\binom{n}{k}\cos^k(x)\sin^{n-k}(x)\sin\left(\frac{1}{2}(n-k)\pi\right)}{\sum_{k=0}^{n}\binom{n}{k}\cos^k(x)\sin^{n-k}(x)\cos\left(\frac{1}{2}(n-k)\pi\right)}$$

Cot:

$$\cot 2\theta = \frac{\cot^2\theta - 1}{2\cot\theta}$$

9.8 TRIPLE ANGLE FORMULAE:

Sine:

$$\sin 3\theta = 3\sin\theta - 4\sin^3\theta$$

Cosine:

$$\cos 3\theta = 4\cos^3\theta - 3\cos\theta$$

Tangent:

$$\tan 3\theta = \frac{3\tan\theta - \tan^3\theta}{1-3\tan^2\theta}$$

Cot:

$$\cot 3\theta = \frac{3\cot\theta - \cot^3\theta}{1-3\cot^2\theta}$$

9.9 HALF ANGLE FORMULAE:

Sine:

$$\sin\frac{\theta}{2} = \pm\sqrt{\frac{1-\cos\theta}{2}}$$

Cosine:

$$\cos\frac{\theta}{2} = \pm\sqrt{\frac{1+\cos\theta}{2}}$$

Tangent:

$$\begin{aligned}\tan\frac{\theta}{2} &= \csc\theta - \cot\theta \\ &= \pm\sqrt{\frac{1-\cos\theta}{1+\cos\theta}} \\ &= \frac{\sin\theta}{1+\cos\theta} \\ &= \frac{1-\cos\theta}{\sin\theta}\end{aligned}$$

Cot:

$$\cot\frac{\theta}{2} = \csc\theta + \cot\theta$$
$$= \pm\sqrt{\frac{1+\cos\theta}{1-\cos\theta}}$$
$$= \frac{\sin\theta}{1-\cos\theta}$$
$$= \frac{1+\cos\theta}{\sin\theta}$$

9.10 POWER REDUCTION:

Sine:

$$\sin^2\theta = \frac{1-\cos 2\theta}{2}$$
$$\sin^3\theta = \frac{3\sin\theta - \sin 3\theta}{4}$$
$$\sin^4\theta = \frac{3 - 4\cos 2\theta + \cos 4\theta}{8}$$
$$\sin^5\theta = \frac{10\sin\theta - 5\sin 3\theta + \sin 5\theta}{16}$$

If n is even:

$$\sin^n\theta = \frac{1}{2^n}\binom{n}{\frac{n}{2}} + \frac{2}{2^n}\sum_{k=0}^{\frac{n}{2}-1}(-1)^{\left(\frac{n}{2}-k\right)}\binom{n}{k}\cos\left((n-2k)\theta\right)$$

If n is odd:

$$\sin^n\theta = \frac{2}{2^n}\sum_{k=0}^{\frac{n-1}{2}}(-1)^{\left(\frac{n-1}{2}-k\right)}\binom{n}{k}\sin\left((n-2k)\theta\right)$$

Cosine:

$$\cos^2\theta = \frac{1+\cos 2\theta}{2}$$
$$\cos^3\theta = \frac{3\cos\theta + \cos 3\theta}{4}$$
$$\cos^4\theta = \frac{3 + 4\cos 2\theta + \cos 4\theta}{8}$$
$$\cos^5\theta = \frac{10\cos\theta + 5\cos 3\theta + \cos 5\theta}{16}$$

If n is even:

$$\cos^n\theta = \frac{2}{2^n}\sum_{k=0}^{\frac{n-1}{2}}\binom{n}{k}\cos\left((n-2k)\theta\right)$$

If n is odd:

$$\cos^n\theta = \frac{1}{2^n}\binom{n}{\frac{n}{2}} + \frac{2}{2^n}\sum_{k=0}^{\frac{n}{2}-1}\binom{n}{k}\cos\left((n-2k)\theta\right)$$

Sine & Cosine:

$$\sin^2\theta\cos^2\theta = \frac{1-\cos 4\theta}{8}$$

$$\sin^3\theta\cos^3\theta = \frac{3\sin 2\theta - \sin 6\theta}{32}$$

$$\sin^4\theta\cos^4\theta = \frac{3-4\cos 4\theta + \cos 8\theta}{128}$$

$$\sin^5\theta\cos^5\theta = \frac{10\sin 2\theta - 5\sin 6\theta + \sin 10\theta}{512}$$

9.11 PRODUCT TO SUM:

$$\cos\theta\cos\varphi = \frac{\cos(\theta-\varphi)+\cos(\theta+\varphi)}{2}$$

$$\sin\theta\sin\varphi = \frac{\cos(\theta-\varphi)-\cos(\theta+\varphi)}{2}$$

$$\sin\theta\cos\varphi = \frac{\sin(\theta+\varphi)+\sin(\theta-\varphi)}{2}$$

$$\cos\theta\sin\varphi = \frac{\sin(\theta+\varphi)-\sin(\theta-\varphi)}{2}$$

9.12 SUM TO PRODUCT:

$$\sin\theta\pm\sin\varphi = 2\sin\left(\frac{\theta\pm\varphi}{2}\right)\cos\left(\frac{\theta\mp\varphi}{2}\right)$$

$$\cos\theta+\cos\varphi = 2\cos\left(\frac{\theta+\varphi}{2}\right)\cos\left(\frac{\theta-\varphi}{2}\right)$$

$$\cos\theta-\cos\varphi = -2\sin\left(\frac{\theta+\varphi}{2}\right)\sin\left(\frac{\theta-\varphi}{2}\right)$$

9.13 HYPERBOLIC EXPRESSIONS:

Hyperbolic sine:

$$\sinh x = \frac{e^x - e^{-x}}{2}$$

Hyperbolic cosine:

$$\cosh x = \frac{e^x + e^{-x}}{2}$$

Hyperbolic tangent:

$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{\frac{1}{2}(e^x - e^{-x})}{\frac{1}{2}(e^x + e^{-x})} = \frac{e^{2x}-1}{e^{2x}+1}$$

Hyperbolic cotangent:

$$\coth x = \frac{\cosh x}{\sinh x} = \frac{\frac{1}{2}(e^x + e^{-x})}{\frac{1}{2}(e^x - e^{-x})} = \frac{e^{2x}+1}{e^{2x}-1}$$

Hyperbolic secant: $$\operatorname{sech} x = \frac{1}{\cosh x} = \frac{2}{e^x + e^{-x}}$$

Hyperbolic cosecant: $$\operatorname{csch} x = \frac{1}{\sinh x} = \frac{2}{e^x - e^{-x}}$$

9.14 HYPERBOLIC RELATIONS:

$$\cosh^2 x - \sinh^2 x = 1$$
$$\sinh(-x) = -\sinh x$$
$$\cosh(-x) = \cosh x$$
$$\tanh(-x) = -\tanh x$$
$$\coth(-x) = -\coth x$$
$$\operatorname{sech}(-x) = \operatorname{sech} x$$
$$\operatorname{csch}(-x) = -\operatorname{csch} x$$
$$\operatorname{arsech} x = \operatorname{arcosh} \frac{1}{x}$$
$$\operatorname{arcsch} x = \operatorname{arsinh} \frac{1}{x}$$
$$\operatorname{arcoth} x = \operatorname{artanh} \frac{1}{x}$$

9.15 MACHIN-LIKE FORMULAE:

Form:

$$\frac{\pi}{4} = \sum_{n}^{N} a_n \arctan \frac{1}{b_n}$$

Formulae:

$$\frac{\pi}{4} = 4 \arctan \frac{1}{5} - \arctan \frac{1}{239}$$
$$\frac{\pi}{4} = \arctan \frac{1}{2} + \arctan \frac{1}{3}$$
$$\frac{\pi}{4} = 2 \arctan \frac{1}{2} - \arctan \frac{1}{7}$$
$$\frac{\pi}{4} = 2 \arctan \frac{1}{3} + \arctan \frac{1}{7}$$
$$\frac{\pi}{4} = 5 \arctan \frac{1}{7} + 2 \arctan \frac{3}{79}$$
$$\frac{\pi}{4} = 12 \arctan \frac{1}{49} + 32 \arctan \frac{1}{57} - 5 \arctan \frac{1}{239} + 12 \arctan \frac{1}{110443}$$
$$\frac{\pi}{4} = 44 \arctan \frac{1}{57} + 7 \arctan \frac{1}{239} - 12 \arctan \frac{1}{682} + 24 \arctan \frac{1}{12943}$$

$$\frac{\pi}{4} = 183\arctan\frac{1}{239} + 32\arctan\frac{1}{1023} - 68\arctan\frac{1}{5832} + 12\arctan\frac{1}{110443} - 12\arctan\frac{1}{4841182} - 100\arctan\frac{1}{6826318}$$

$$\frac{\pi}{4} = 183\arctan\frac{1}{239} + 32\arctan\frac{1}{1023} - 68\arctan\frac{1}{5832} + 12\arctan\frac{1}{113021} - 100\arctan\frac{1}{6826318} - 12\arctan\frac{1}{33366019650} + 12\arctan\frac{1}{43599522992503626068}$$

Identities:

$\arctan x + \arctan y = \arctan\frac{x+y}{1-xy}$ for $xy < 1$,

$\arctan x - \arctan y = \arctan\frac{x-y}{1+xy}$ for $xy > -1$,

$\arctan\frac{a}{b} + \arctan\frac{c}{d} = \arctan\frac{ad+bc}{bd-ac}$ for $\frac{ac}{bd} < 1$,

$\arctan\frac{a}{b} - \arctan\frac{c}{d} = \arctan\frac{ad-bc}{bd+ac}$ for $\frac{ac}{bd} > -1$.

9.16 SPHERICAL TRIANGLE IDENTITIES:

$$\frac{\sin\left(\frac{1}{2}(A-B)\right)}{\sin\left(\frac{1}{2}(A+B)\right)} = \frac{\tan\left(\frac{1}{2}(a-b)\right)}{\tan\left(\frac{1}{2}c\right)}$$

$$\frac{\sin\left(\frac{1}{2}(a-b)\right)}{\sin\left(\frac{1}{2}(a+b)\right)} = \frac{\tan\left(\frac{1}{2}(A-B)\right)}{\cot\left(\frac{1}{2}c\right)}$$

$$\frac{\cos\left(\frac{1}{2}(A-B)\right)}{\cos\left(\frac{1}{2}(A+B)\right)} = \frac{\tan\left(\frac{1}{2}(a+b)\right)}{\tan\left(\frac{1}{2}c\right)}$$

$$\frac{\cos\left(\frac{1}{2}(a-b)\right)}{\cos\left(\frac{1}{2}(a+b)\right)} = \frac{\tan\left(\frac{1}{2}(A+B)\right)}{\cot\left(\frac{1}{2}c\right)}$$

9.17 ABBREVIATIONS (9.1-9.16)

A=Angle ‘A’

a=side ‘a’
B=Angle ‘B’
b=side ‘b’
B=Angle ‘B’
c=side ‘c’

PART 10: EXPONENTIALS & LOGARITHIMS

10.1 FUNDAMENTAL THEORY:

$$e^{\ln(x)} = x \qquad \text{if } x > 0$$

$e = 2.71828182845904523536028747135266249775724709370000$

$$\ln(e^x) = x.$$

$$\ln x = \int_1^x \frac{1}{t} dt$$

$$x = b^a \iff \log_b x = a$$

10.2 EXPONENTIAL IDENTITIES:

$$x^a x^b = x^{a+b}$$

$$\frac{x^a}{x^b} = x^{a-b}$$

$$x^a y^a = (xy)^a$$

$$(x^a)^b = x^{ab} = (x^b)^a$$

$$x^{\frac{a}{b}} = \sqrt[b]{x^a}$$

$$x^{-a} = \frac{1}{x^a}$$

10.3 LOG IDENTITIES:

$$\log_b(1) = 0$$

$$\log_b(b) = 1$$

$$\log_b(xy) = \log_b(x) + \log_b(y)$$

$$\log_b\left(\frac{x}{y}\right) = \log_b(x) - \log_b(y)$$

$$\log_b(x^d) = d\log_b(x)$$

$$\log_b x = \frac{\log_k x}{\log_k b}.$$

$$\log_b\left(\sqrt[y]{x}\right) = \frac{\log_b(x)}{y}$$

$$x^{\log_b(y)} = y^{\log_b(x)}$$

$$c\log_b(x) + d\log_b(y) = \log_b(x^c y^d)$$

$$b^{\log_b(x)} = x \text{ because } \operatorname{antilog}_b(\log_b(x)) = x$$

$$\log_b(b^x) = x \text{ because } \log_b(\operatorname{antilog}_b(x)) = x$$

$$\log(\log(c^d)) = \log(\log(c)) + \log(d)$$

$$\log(\log(\sqrt[d]{c})) = \log(\log(c)) - \log(d)$$

$$\log_b(a+c) = \log_b a + \log_b\left(1 + \frac{c}{a}\right)$$

$$\log_b(a-c) = \log_b a + \log_b\left(1 - \frac{c}{a}\right)$$

$$x^{\frac{\log(\log(x))}{\log(x)}} = \log(x)$$

$$\int \log_a x \, dx = x(\log_a x - \log_a e) + C$$
$$\ln(a \times 10^n) = \ln a + n \ln 10.$$

10.4 LAWS FOR LOG TABLES:

$$xy = b^{\log_b(x)} b^{\log_b(y)} = b^{\log_b(x)+\log_b(y)} \Rightarrow \log_b(xy) = \log_b(b^{\log_b(x)+\log_b(y)}) = \log_b(x)+\log_b(y)$$

$$x^y = (b^{\log_b(x)})^y = b^{y \log_b(x)} \Rightarrow \log_b(x^y) = y \log_b(x)$$

$$\log_b\left(\frac{x}{y}\right) = \log_b(xy^{-1}) = \log_b(x) + \log_b(y^{-1}) = \log_b(x) - \log_b(y)$$

$$\log_b(\sqrt[y]{x}) = \log_b(x^{\frac{1}{y}}) = \frac{1}{y}\log_b(x)$$

10.5 COMPLEX NUMBERS:

$$\log(z) = \ln|z| + i\arg(z) = \ln(r) + i(\theta + 2\pi k)$$
$$\operatorname{Log}(z_1) + \operatorname{Log}(z_2) = \operatorname{Log}(z_1 z_2) \pmod{2\pi i}$$
$$\operatorname{Log}(z_1) - \operatorname{Log}(z_2) = \operatorname{Log}(z_1/z_2) \pmod{2\pi i}$$
$$z_1{}^{z_2} = e^{z_2 \operatorname{Log}(z_1)}$$
$$\operatorname{Log}(z_1{}^{z_2}) = z_2 \operatorname{Log}(z_1) \pmod{2\pi i}$$

10.6 LIMITS INVOLVING LOGARITHMIC TERMS

$$\lim_{x \to 0^+} \log_a x = -\infty \quad \text{if } a > 1$$
$$\lim_{x \to 0^+} \log_a x = \infty \quad \text{if } a < 1$$
$$\lim_{x \to \infty} \log_a x = \infty \quad \text{if } a > 1$$
$$\lim_{x \to \infty} \log_a x = -\infty \quad \text{if } a < 1$$
$$\lim_{x \to 0^+} x^b \log_a x = 0 \quad \text{if } b > 0$$
$$\lim_{x \to \infty} \frac{1}{x^b} \log_a x = 0 \quad \text{if } b > 0$$

PART 11: COMPLEX NUMBERS

11.1 GENERAL:

Fundamental: $i^2 = -1$

Standard Form: $z = a + bi$

Polar Form: $z = rcis\theta = r(\cos\theta + i\sin\theta)$

Argument: $\arg(z) = \theta$, where $\tan\theta = \frac{b}{a}$

Modulus: $\text{mod}(z) = r = |z| = |a+bi| = \sqrt{a^2+b^2}$

Conjugate: $\bar{z} = a - bi$

Exponential: $z = r \cdot e^{i\theta}$

De Moivre's Formula:

$$z = rcis\theta$$

$$z^{\frac{1}{n}} = r^{\frac{1}{n}} cis\left(\frac{\theta + 2k\pi}{n}\right), \text{ k=0,1,…,(n-1)}$$

Euler's Identity: $e^{i\pi} + 1 = 0$ (Special Case when n=2)

$$\sum_{k=0}^{n-1} e^{\frac{2i\pi k}{n}} = 0 \quad \text{(Generally)}$$

11.2 OPERATIONS:

Addition: (a + bi) + (c + di) = (a + c) + (b + d)i

Subtraction: (a + bi) - (c + di) = (a - c) + (b - d)i.

Multiplication: $(a + bi)(c + di) = ac + bci + adi + bdi^2 = (ac - bd) + (bc + ad)i.$

Division: $\frac{(a+bi)}{(c+di)} = \frac{(a+bi)(c-di)}{(c+di)(c-di)} = \frac{ac + bci - adi + bd}{(c+di)(c-di)} = \left(\frac{ac+bd}{c^2+d^2}\right) + \left(\frac{bc-ad}{c^2+d^2}\right)i.$

Sum of Squares: $a^2 + b^2 = (a+bi)(a-bi)$

11.3 IDENTITIES:

Exponential: $e^{i\theta} = \cos(\theta) + i\sin(\theta)$

Logarithmic: $\log(x+iy) = \frac{1}{2}\ln(x^2+y^2) + i\arg(x+iy),$

Trigonometric:

$$\sin(x) = \frac{e^{ix} - e^{-ix}}{2i}$$

$$\cos(x) = \frac{e^{ix} + e^{-ix}}{2}$$

$$\tan(x) = \frac{e^{ix} - e^{-ix}}{i(e^{ix} + e^{-ix})} = \frac{\sin(x)}{\cos(x)}$$

$$sin(a+bi) = sin(a)cosh(b) + cos(a)sinh(b)i$$

$$cos(a+bi) = cos(a)cosh(b) - sin(a)sinh(b)i$$

Hyperbolic:

$$e^{ix} = \cos x + i\ \sin x$$
$$e^{-ix} = \cos x - i\ \sin x$$
$$\cosh ix = \frac{1}{2}(e^{ix} + e^{-ix}) = \cos x$$
$$\sinh ix = \frac{1}{2}(e^{ix} - e^{-ix}) = i \sin x$$
$$\tanh ix = i \tan x$$
$$\cosh x = \cos ix$$
$$\sinh x = -i \sin ix$$
$$\tanh x = -i \tan ix$$

PART 12: DIFFERENTIATION

For Differential Equations, see Functions

12.1 GENERAL RULES:

Plus Or Minus:

$$y = f_{(x)} \pm g_{(x)} \pm h_{(x)} \ldots$$

$$y' = f'_{(x)} \pm g'_{(x)} \pm h'_{(x)} \ldots$$

Product Rule:

$$y = uv$$

$$y' = u'v + uv'$$

Quotient Rule:

$$y = \frac{u}{v}$$

$$y' = \frac{u'v - uv'}{v^2}$$

Power Rule:

$$y = \left(f_{(x)}\right)^n$$

$$y' = n\left(f_{(x)}\right)^{n-1} \times f'_{(x)}$$

Chain Rule:

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dv} \times \frac{dv}{dx}$$

Blob Rule:

$$y = e^{f_{(x)}}$$

$$y' = f'_{(x)} \times e^{f_{(x)}}$$

Base A Log:

$$y = \log_a f_{(x)}$$

$$y' = \frac{f'_{(x)}}{f_{(x)} \times \ln(a)}$$

Natural Log:

$$y = a \ln\left(f_{(x)}\right)$$

$$y' = a \times \frac{f'_{(x)}}{f_{(x)}}$$

Exponential (X):

$$y = k^x$$

$$y' = \ln k \times k^x$$

First Principles:

$$f'_{(x)} = \lim_{h \longrightarrow 0} \left(\frac{f_{(x+h)} - f_{(x)}}{h} \right)$$

Angle of intersection between two curves:

Two curves $y = f_1(x)$ & $y = f_2(x)$ intersect at x_0

$$\alpha = \tan^{-1}\left(\frac{f'_2(x_0) - f'_1(x_0)}{1 + f'_2(x_0) \times f'_1(x_0)}\right)$$

12.2 EXPONETIAL FUNCTIONS:

$$\frac{d}{dx}e^x = e^x.$$
$$\frac{d}{dx}a^x = (\ln a)a^x.$$

12.3 LOGARITHMIC FUNCTIONS:

$$\frac{d}{dx}\ln x = \frac{1}{x} = \frac{1}{x\ln e}, \qquad x > 0$$
$$\frac{d}{dx}\log_b x = \frac{1}{x\ln b}, \qquad b > 0, b \neq 1$$
$$\frac{d}{dx}\log_b(x) = \frac{d}{dx}\frac{\ln(x)}{\ln(b)} = \frac{1}{x\ln(b)} = \frac{\log_b(e)}{x}.$$

12.4 TRIGONOMETRIC FUNCTIONS:

$$(\sin(x))' = \cos(x)$$
$$(\cos(x))' = -\sin(x)$$
$$(\tan(x))' = \left(\frac{\sin(x)}{\cos(x)}\right)' = \frac{\cos^2(x) + \sin^2(x)}{\cos^2(x)} = \frac{1}{\cos^2(x)} = \sec^2(x)$$
$$(\cot(x))' = \left(\frac{\cos(x)}{\sin(x)}\right)' = \frac{-\sin^2(x) - \cos^2(x)}{\sin^2(x)} = -(1 + \cot^2(x)) = -\csc^2(x)$$
$$(\sec(x))' = \left(\frac{1}{\cos(x)}\right)' = \frac{\sin(x)}{\cos^2(x)} = \frac{1}{\cos(x)}.\frac{\sin(x)}{\cos(x)} = \sec(x)\tan(x)$$
$$(\csc(x))' = \left(\frac{1}{\sin(x)}\right)' = -\frac{\cos(x)}{\sin^2(x)} = -\frac{1}{\sin(x)}.\frac{\cos(x)}{\sin(x)} = -\csc(x)\cot(x)$$
$$(\arcsin(x))' = \frac{1}{\sqrt{1 - x^2}}$$
$$(\arccos(x))' = \frac{-1}{\sqrt{1 - x^2}}$$
$$(\arctan(x))' = \frac{1}{x^2 + 1}$$

$$(arc\cot(x))' = \frac{-1}{x^2+1}$$

$$(arc\sec(x))' = \frac{1}{x^2\sqrt{1-\frac{1}{x^2}}}$$

$$(arc\csc(x))' = \frac{-1}{x^2\sqrt{1-\frac{1}{x^2}}}$$

12.5 HYPERBOLIC FUNCTIONS:

$$\frac{d}{dx}\sinh x = \cosh x$$

$$\frac{d}{dx}\cosh x = \sinh x$$

$$\frac{d}{dx}\tanh x = 1 - \tanh^2 x = \text{sech}^2 x = 1/\cosh^2 x$$

$$\frac{d}{dx}\coth x = 1 - \coth^2 x = -\text{csch}^2 x = -1/\sinh^2 x$$

$$\frac{d}{dx}\ \text{csch}\, x = -\coth x\ \text{csch}\, x$$

$$\frac{d}{dx}\ \text{sech}\, x = -\tanh x\ \text{sech}\, x$$

$$\frac{d}{dx}\left(\sinh^{-1} x\right) = \frac{1}{\sqrt{x^2+1}}$$

$$\frac{d}{dx}\left(\cosh^{-1} x\right) = \frac{1}{\sqrt{x^2-1}}$$

$$\frac{d}{dx}\left(\tanh^{-1} x\right) = \frac{1}{1-x^2}$$

$$(arc\coth(x))' = \frac{-1}{x^2-0}$$

$$(arc\sec h(x))' = \frac{-1}{\sqrt{x^2-x^4}}$$

$$(arc\csc h(x))' = \frac{-1}{\sqrt{x^2+x^4}}$$

12.5 PARTIAL DIFFERENTIATION:

First Principles:

$$\frac{\partial f}{\partial x}(a,b) = \frac{d(f(x,b))}{dx}\Big|_{x=a} = \lim_{h\to 0}\frac{f(a+h,b)-f(a,b)}{h}$$

$$\frac{\partial f}{\partial y}(a,b) = \frac{d(f(a,y))}{dy}|_{y=b} = \lim_{h \to 0} \frac{f(a,b+h) - f(a,b)}{h}$$

ie:

$$\frac{\partial f}{\partial x} = \lim_{h \to 0} \frac{f(x+h,y) - f(x,y)}{h}$$

$$\frac{\partial f}{\partial y} = \lim_{h \to 0} \frac{f(x,y+h) - f(x,y)}{h}$$

Gradient:

$$\nabla f = \left[\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right]$$

Total Differential:

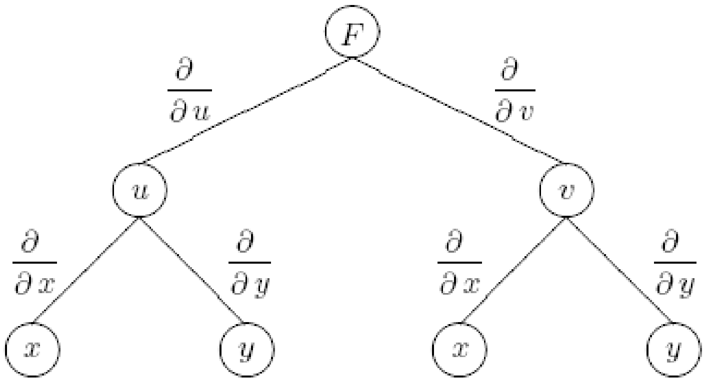
$$df = f_x(x,y)\,dx + f_y(x,y)\,dy = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$$

Chain Rule:

Case 1. Suppose $F = F(u,v)$ and $u = u(x,y)$, $v = v(x,y)$. Then F is also a function of x and y and

$$\frac{\partial F}{\partial x} = \frac{\partial F}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial F}{\partial v}\frac{\partial v}{\partial x}$$

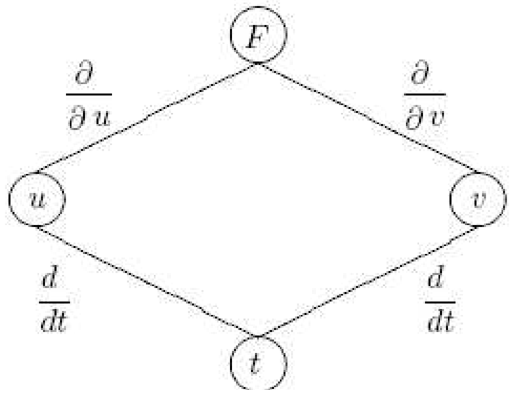
$$\frac{\partial F}{\partial y} = \frac{\partial F}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial F}{\partial v}\frac{\partial v}{\partial y}$$



Case 2. Suppose $F = F(u,v)$ and $u = u(t)$, $v = v(t)$. Then F is a function of t only and

$$\frac{dF}{dt} = \frac{\partial F}{\partial u}\frac{du}{dt} + \frac{\partial F}{\partial v}\frac{dv}{dt}$$

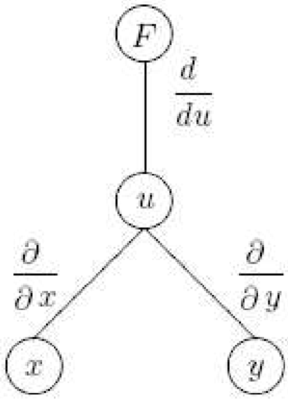
(Note the 'full' derivative here)



Case 3. $F = F(u)$, $u = u(x, y)$. Then F is a function of x and y and

$$\frac{\partial F}{\partial x} = \frac{dF}{du}\frac{\partial u}{\partial x}$$

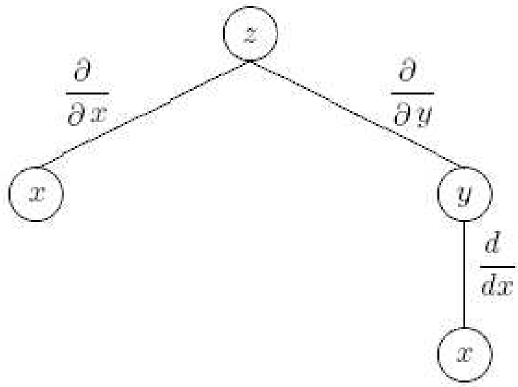
$$\frac{\partial F}{\partial y} = \frac{dF}{du}\frac{\partial u}{\partial y}$$



Implicit Differentiation:

Consider $z = F(x, y)$ where we assume the y is a function of x, *i.e.* $y = y(x)$. Then z is ultimately a function of x only and the chain rule tells us

$$\frac{dz}{dx} = \frac{\partial F}{\partial x} + \frac{\partial F}{\partial y}\frac{dy}{dx}.$$



Therefore,

$$\boxed{\frac{dy}{dx} = -\frac{\partial F}{\partial x}\Big/\frac{\partial F}{\partial y} = -\frac{F_x}{F_y}}$$

Higher Order Derivatives:

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) \equiv \frac{\partial^2 f}{\partial x^2} \equiv f_{xx}$$

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) \equiv \frac{\partial^2 f}{\partial y^2} \equiv f_{yy}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) \equiv \frac{\partial^2 f}{\partial x\,\partial y} \equiv f_{yx}$$

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) \equiv \frac{\partial^2 f}{\partial y\,\partial x} \equiv f_{xy}$$

PART 13: INTEGRATION

13.1 GENERAL RULES:

Power Rule:

$$\int f'_{(x)}\left[f_{(x)}\right]^n dx = \frac{\left[f_{(x)}\right]^{n+1}}{n+1} + C$$

$$a\int f'_{(x)}\left[f_{(x)}\right]^n dx = a\frac{\left[f_{(x)}\right]^{n+1}}{n+1} + C$$

By Parts:

$$\int u\,dv = uv - \int v\,du$$

Constants:

$$\int_0^{f(x)} k\,dt = kf(x)$$

13.2 RATIONAL FUNCTIONS:

$$\int (ax+b)^n dx = \frac{(ax+b)^{n+1}}{a(n+1)} + C \qquad (\text{for } n \neq -1)$$

$$\int \frac{c}{ax+b} dx = \frac{c}{a}\ln|ax+b| + C$$

$$\int x(ax+b)^n dx = \frac{a(n+1)x-b}{a^2(n+1)(n+2)}(ax+b)^{n+1} + C \qquad (\text{for } n \notin \{-1,-2\})$$

$$\int \frac{x}{ax+b} dx = \frac{x}{a} - \frac{b}{a^2}\ln|ax+b| + C$$

$$\int \frac{x}{(ax+b)^2} dx = \frac{b}{a^2(ax+b)} + \frac{1}{a^2}\ln|ax+b| + C$$

$$\int \frac{x}{(ax+b)^n} dx = \frac{a(1-n)x-b}{a^2(n-1)(n-2)(ax+b)^{n-1}} + C \qquad (\text{for } n \notin \{1,2\})$$

$$\int \frac{x^2}{ax+b} dx = \frac{b^2\ln(|ax+b|)}{a^3} + \frac{ax^2-2bx}{2a^2} + C$$

$$\int \frac{x^2}{(ax+b)^2} dx = \frac{1}{a^3}\left(ax - 2b\ln|ax+b| - \frac{b^2}{ax+b}\right) + C$$

$$\int \frac{x^2}{(ax+b)^3} dx = \frac{1}{a^3}\left(\ln|ax+b| + \frac{2b}{ax+b} - \frac{b^2}{2(ax+b)^2}\right) + C$$

$$\int \frac{x^2}{(ax+b)^n} dx = \frac{1}{a^3}\left(-\frac{(ax+b)^{3-n}}{(n-3)} + \frac{2b(ax+b)^{2-n}}{(n-2)} - \frac{b^2(ax+b)^{1-n}}{(n-1)}\right) + C \qquad (\text{for } n \notin \{1,2,3\})$$

$$\int \frac{1}{x(ax+b)} dx = -\frac{1}{b}\ln\left|\frac{ax+b}{x}\right| + C$$

$$\int \frac{1}{x^2(ax+b)} dx = -\frac{1}{bx} + \frac{a}{b^2}\ln\left|\frac{ax+b}{x}\right| + C$$

$$\int \frac{1}{x^2(ax+b)^2} dx = -a\left(\frac{1}{b^2(ax+b)} + \frac{1}{ab^2x} - \frac{2}{b^3}\ln\left|\frac{ax+b}{x}\right|\right) + C$$

$$\int \frac{1}{x^2+a^2}dx = \frac{1}{a}\arctan\frac{x}{a} + C$$

$$\int \frac{1}{x^2-a^2}dx = \begin{cases} -\frac{1}{a}\operatorname{arctanh}\frac{x}{a} = \frac{1}{2a}\ln\frac{a-x}{a+x} + C & (\text{for } |x| < |a|) \\ -\frac{1}{a}\operatorname{arccoth}\frac{x}{a} = \frac{1}{2a}\ln\frac{x-a}{x+a} + C & (\text{for } |x| > |a|) \end{cases}$$

For $a \neq 0:$

$$\int \frac{1}{ax^2+bx+c}dx = \begin{cases} \frac{2}{\sqrt{4ac-b^2}}\arctan\frac{2ax+b}{\sqrt{4ac-b^2}} + C & (\text{for } 4ac-b^2>0) \\ -\frac{2}{\sqrt{b^2-4ac}}\operatorname{arctanh}\frac{2ax+b}{\sqrt{b^2-4ac}} + C = \frac{1}{\sqrt{b^2-4ac}}\ln\left|\frac{2ax+b-\sqrt{b^2-4ac}}{2ax+b+\sqrt{b^2-4ac}}\right| + C & (\text{for } 4ac-b^2<0) \\ -\frac{2}{2ax+b} + C & (\text{for } 4ac-b^2=0) \end{cases}$$

$$\int \frac{x}{ax^2+bx+c}dx = \frac{1}{2a}\ln\left|ax^2+bx+c\right| - \frac{b}{2a}\int\frac{dx}{ax^2+bx+c} + C$$

$$\int \frac{mx+n}{ax^2+bx+c}dx = \begin{cases} \frac{m}{2a}\ln\left|ax^2+bx+c\right| + \frac{2an-bm}{a\sqrt{4ac-b^2}}\arctan\frac{2ax+b}{\sqrt{4ac-b^2}} + C & (\text{for } 4ac-b^2>0) \\ \frac{m}{2a}\ln\left|ax^2+bx+c\right| - \frac{2an-bm}{a\sqrt{b^2-4ac}}\operatorname{arctanh}\frac{2ax+b}{\sqrt{b^2-4ac}} + C & (\text{for } 4ac-b^2<0) \\ \frac{m}{2a}\ln\left|ax^2+bx+c\right| - \frac{2an-bm}{a(2ax+b)} + C & (\text{for } 4ac-b^2=0) \end{cases}$$

$$\int \frac{1}{(ax^2+bx+c)^n}dx = \frac{2ax+b}{(n-1)(4ac-b^2)(ax^2+bx+c)^{n-1}} + \frac{(2n-3)2a}{(n-1)(4ac-b^2)}\int\frac{1}{(ax^2+bx+c)^{n-1}}dx + C$$

$$\int \frac{x}{(ax^2+bx+c)^n}dx = -\frac{bx+2c}{(n-1)(4ac-b^2)(ax^2+bx+c)^{n-1}} - \frac{b(2n-3)}{(n-1)(4ac-b^2)}\int\frac{1}{(ax^2+bx+c)^{n-1}}dx + C$$

$$\int \frac{1}{x(ax^2+bx+c)}dx = \frac{1}{2c}\ln\left|\frac{x^2}{ax^2+bx+c}\right| - \frac{b}{2c}\int\frac{1}{ax^2+bx+c}dx + C$$

$$\int \frac{dx}{x^{2^n}+1} = \sum_{k=1}^{2^{n-1}}\left\{\frac{1}{2^{n-1}}\left[\sin\left(\frac{(2k-1)\pi}{2^n}\right)\arctan\left[\left(x-\cos\left(\frac{(2k-1)\pi}{2^n}\right)\right)\csc\left(\frac{(2k-1)\pi}{2^n}\right)\right]\right] - \frac{1}{2^n}\left[\cos\left(\frac{(2k-1)\pi}{2^n}\right)\ln\left|x^2-2x\cos\left(\frac{(2k-1)\pi}{2^n}\right)+1\right|\right]\right\}$$

13.3 TRIGONOMETRIC FUNCTIONS (SINE):

$$\int \sin ax\; dx = -\frac{1}{a}\cos ax + C$$

$$\int \sin^2 ax\; dx = \frac{x}{2} - \frac{1}{4a}\sin 2ax + C = \frac{x}{2} - \frac{1}{2a}\sin ax\cos ax + C$$

$$\int \sin^n ax\; dx = -\frac{\sin^{n-1}ax\cos ax}{na} + \frac{n-1}{n}\int\sin^{n-2}ax\; dx \qquad (\text{for } n>0)$$

$$\int x\sin^2 ax\; dx = \frac{x^2}{4} - \frac{x}{4a}\sin 2ax - \frac{1}{8a^2}\cos 2ax + C$$

$$\int x^2\sin^2 ax\; dx = \frac{x^3}{6} - \left(\frac{x^2}{4a} - \frac{1}{8a^3}\right)\sin 2ax - \frac{x}{4a^2}\cos 2ax + C$$

$$\int \sin b_1x\sin b_2x\; dx = \frac{\sin((b_1-b_2)x)}{2(b_1-b_2)} - \frac{\sin((b_1+b_2)x)}{2(b_1+b_2)} + C \qquad (\text{for } |b_1| \neq |b_2|)$$

$$\int \frac{dx}{\sin ax} = \frac{1}{a} \ln \left| \tan \frac{ax}{2} \right| + C$$

$$\int \frac{dx}{\sin^n ax} = \frac{\cos ax}{a(1-n)\sin^{n-1} ax} + \frac{n-2}{n-1} \int \frac{dx}{\sin^{n-2} ax} \qquad (\text{for } n > 1)$$

$$\int x \sin ax \; dx = \frac{\sin ax}{a^2} - \frac{x \cos ax}{a} + C$$

$$\int x^n \sin ax \; dx = -\frac{x^n}{a} \cos ax + \frac{n}{a} \int x^{n-1} \cos ax \; dx \qquad (\text{for } n > 0)$$

$$\int_{\frac{-a}{2}}^{\frac{a}{2}} x^2 \sin^2 \frac{n\pi x}{a} \; dx = \frac{a^3(n^2\pi^2 - 6)}{24n^2\pi^2} \qquad (\text{for } n = 2, 4, 6...)$$

$$\int \frac{\sin ax}{x} dx = \sum_{n=0}^{\infty} (-1)^n \frac{(ax)^{2n+1}}{(2n+1) \cdot (2n+1)!} + C$$

$$\int \frac{\sin ax}{x^n} dx = -\frac{\sin ax}{(n-1)x^{n-1}} + \frac{a}{n-1} \int \frac{\cos ax}{x^{n-1}} dx$$

$$\int \frac{dx}{1 \pm \sin ax} = \frac{1}{a} \tan \left(\frac{ax}{2} \mp \frac{\pi}{4} \right) + C$$

$$\int \frac{x \; dx}{1 + \sin ax} = \frac{x}{a} \tan \left(\frac{ax}{2} - \frac{\pi}{4} \right) + \frac{2}{a^2} \ln \left| \cos \left(\frac{ax}{2} - \frac{\pi}{4} \right) \right| + C$$

$$\int \frac{x \; dx}{1 - \sin ax} = \frac{x}{a} \cot \left(\frac{\pi}{4} - \frac{ax}{2} \right) + \frac{2}{a^2} \ln \left| \sin \left(\frac{\pi}{4} - \frac{ax}{2} \right) \right| + C$$

$$\int \frac{\sin ax \; dx}{1 \pm \sin ax} = \pm x + \frac{1}{a} \tan \left(\frac{\pi}{4} \mp \frac{ax}{2} \right) + C$$

13.4 TRIGONOMETRIC FUNCTIONS (COSINE):

$$\int \cos ax \; dx = \frac{1}{a} \sin ax + C$$

$$\int \cos^2 ax \; dx = \frac{x}{2} + \frac{1}{4a} \sin 2ax + C = \frac{x}{2} + \frac{1}{2a} \sin ax \cos ax + C$$

$$\int \cos^n ax \; dx = \frac{\cos^{n-1} ax \sin ax}{na} + \frac{n-1}{n} \int \cos^{n-2} ax \; dx \qquad (\text{for } n > 0)$$

$$\int x \cos ax \; dx = \frac{\cos ax}{a^2} + \frac{x \sin ax}{a} + C$$

$$\int x^2 \cos^2 ax \; dx = \frac{x^3}{6} + \left(\frac{x^2}{4a} - \frac{1}{8a^3} \right) \sin 2ax + \frac{x}{4a^2} \cos 2ax + C$$

$$\int x^n \cos ax \; dx = \frac{x^n \sin ax}{a} - \frac{n}{a} \int x^{n-1} \sin ax \; dx$$

$$\int \frac{\cos ax}{x} dx = \ln |ax| + \sum_{k=1}^{\infty} (-1)^k \frac{(ax)^{2k}}{2k \cdot (2k)!} + C$$

$$\int \frac{\cos ax}{x^n} dx = -\frac{\cos ax}{(n-1)x^{n-1}} - \frac{a}{n-1} \int \frac{\sin ax}{x^{n-1}} dx \qquad (\text{for } n \neq 1)$$

$$\int \frac{dx}{\cos ax} = \frac{1}{a}\ln\left|\tan\left(\frac{ax}{2}+\frac{\pi}{4}\right)\right| + C$$
$$\int \frac{dx}{\cos^n ax} = \frac{\sin ax}{a(n-1)\cos^{n-1} ax} + \frac{n-2}{n-1}\int \frac{dx}{\cos^{n-2} ax} \qquad (\text{for } n > 1)$$
$$\int \frac{dx}{1+\cos ax} = \frac{1}{a}\tan\frac{ax}{2} + C$$
$$\int \frac{dx}{1-\cos ax} = -\frac{1}{a}\cot\frac{ax}{2} + C$$
$$\int \frac{x\,dx}{1+\cos ax} = \frac{x}{a}\tan\frac{ax}{2} + \frac{2}{a^2}\ln\left|\cos\frac{ax}{2}\right| + C$$
$$\int \frac{x\,dx}{1-\cos ax} = -\frac{x}{a}\cot\frac{ax}{2} + \frac{2}{a^2}\ln\left|\sin\frac{ax}{2}\right| + C$$
$$\int \frac{\cos ax\,dx}{1+\cos ax} = x - \frac{1}{a}\tan\frac{ax}{2} + C$$
$$\int \frac{\cos ax\,dx}{1-\cos ax} = -x - \frac{1}{a}\cot\frac{ax}{2} + C$$
$$\int \cos a_1x \cos a_2x\,dx = \frac{\sin(a_1-a_2)x}{2(a_1-a_2)} + \frac{\sin(a_1+a_2)x}{2(a_1+a_2)} + C \qquad (\text{for } |a_1| \neq |a_2|)$$

13.5 TRIGONOMETRIC FUNCTIONS (TANGENT):

$$\int \tan ax\,dx = -\frac{1}{a}\ln|\cos ax| + C = \frac{1}{a}\ln|\sec ax| + C$$
$$\int \tan^n ax\,dx = \frac{1}{a(n-1)}\tan^{n-1} ax - \int \tan^{n-2} ax\,dx \qquad (\text{for } n \neq 1)$$
$$\int \frac{dx}{q\tan ax + p} = \frac{1}{p^2+q^2}\left(px + \frac{q}{a}\ln|q\sin ax + p\cos ax|\right) + C \qquad (\text{for } p^2+q^2 \neq 0)$$
$$\int \frac{dx}{\tan ax} = \frac{1}{a}\ln|\sin ax| + C$$
$$\int \frac{dx}{\tan ax + 1} = \frac{x}{2} + \frac{1}{2a}\ln|\sin ax + \cos ax| + C$$
$$\int \frac{dx}{\tan ax - 1} = -\frac{x}{2} + \frac{1}{2a}\ln|\sin ax - \cos ax| + C$$
$$\int \frac{\tan ax\,dx}{\tan ax + 1} = \frac{x}{2} - \frac{1}{2a}\ln|\sin ax + \cos ax| + C$$
$$\int \frac{\tan ax\,dx}{\tan ax - 1} = \frac{x}{2} + \frac{1}{2a}\ln|\sin ax - \cos ax| + C$$

13.6 TRIGONOMETRIC FUNCTIONS (SECANT):

$$\int \sec ax\,dx = \frac{1}{a}\ln|\sec ax + \tan ax| + C$$
$$\int \sec^2 x\,dx = \tan x + C$$

$$\int \sec^n ax\,dx = \frac{\sec^{n-1} ax \sin ax}{a(n-1)} + \frac{n-2}{n-1}\int \sec^{n-2} ax\,dx \qquad (\text{for } n \neq 1)$$

$$\int \sec^n x\,dx = \frac{\sec^{n-2} x \tan x}{n-1} + \frac{n-2}{n-1}\int \sec^{n-2} x\,dx$$

$$\int \frac{dx}{\sec x + 1} = x - \tan\frac{x}{2} + C$$

$$\int \frac{dx}{\sec x - 1} = -x - \cot\frac{x}{2} + C$$

13.7 TRIGONOMETRIC FUNCTIONS (COTANGENT):

$$\int \cot ax\; dx = \frac{1}{a}\ln|\sin ax| + C$$

$$\int \cot^n ax\; dx = -\frac{1}{a(n-1)}\cot^{n-1} ax - \int \cot^{n-2} ax\; dx \qquad (\text{for } n \neq 1)$$

$$\int \frac{dx}{1+\cot ax} = \int \frac{\tan ax\; dx}{\tan ax + 1}$$

$$\int \frac{dx}{1-\cot ax} = \int \frac{\tan ax\; dx}{\tan ax - 1}$$

13.8 TRIGONOMETRIC FUNCTIONS (SINE & COSINE):

$$\int \frac{dx}{\cos ax \pm \sin ax} = \frac{1}{a\sqrt{2}}\ln\left|\tan\left(\frac{ax}{2} \pm \frac{\pi}{8}\right)\right| + C$$

$$\int \frac{dx}{(\cos ax \pm \sin ax)^2} = \frac{1}{2a}\tan\left(ax \mp \frac{\pi}{4}\right) + C$$

$$\int \frac{dx}{(\cos x + \sin x)^n} = \frac{1}{n-1}\left(\frac{\sin x - \cos x}{(\cos x + \sin x)^{n-1}} - 2(n-2)\int \frac{dx}{(\cos x + \sin x)^{n-2}}\right)$$

$$\int \frac{\cos ax\; dx}{\cos ax + \sin ax} = \frac{x}{2} + \frac{1}{2a}\ln|\sin ax + \cos ax| + C$$

$$\int \frac{\cos ax\; dx}{\cos ax - \sin ax} = \frac{x}{2} - \frac{1}{2a}\ln|\sin ax - \cos ax| + C$$

$$\int \frac{\sin ax\; dx}{\cos ax + \sin ax} = \frac{x}{2} - \frac{1}{2a}\ln|\sin ax + \cos ax| + C$$

$$\int \frac{\sin ax\; dx}{\cos ax - \sin ax} = -\frac{x}{2} - \frac{1}{2a}\ln|\sin ax - \cos ax| + C$$

$$\int \frac{\cos ax\; dx}{\sin ax(1+\cos ax)} = -\frac{1}{4a}\tan^2\frac{ax}{2} + \frac{1}{2a}\ln\left|\tan\frac{ax}{2}\right| + C$$

$$\int \frac{\cos ax\; dx}{\sin ax(1-\cos ax)} = -\frac{1}{4a}\cot^2\frac{ax}{2} - \frac{1}{2a}\ln\left|\tan\frac{ax}{2}\right| + C$$

$$\int \frac{\sin ax\ dx}{\cos ax(1+\sin ax)} = \frac{1}{4a}\cot^2\left(\frac{ax}{2}+\frac{\pi}{4}\right) + \frac{1}{2a}\ln\left|\tan\left(\frac{ax}{2}+\frac{\pi}{4}\right)\right| + C$$

$$\int \frac{\sin ax\ dx}{\cos ax(1-\sin ax)} = \frac{1}{4a}\tan^2\left(\frac{ax}{2}+\frac{\pi}{4}\right) - \frac{1}{2a}\ln\left|\tan\left(\frac{ax}{2}+\frac{\pi}{4}\right)\right| + C$$

$$\int \sin ax \cos ax\ dx = -\frac{1}{2a}\cos^2 ax + C$$

$$\int \sin a_1 x \cos a_2 x\ dx = -\frac{\cos((a_1-a_2)x)}{2(a_1-a_2)} - \frac{\cos((a_1+a_2)x)}{2(a_1+a_2)} + C \qquad (\text{for } |a_1| \neq |a_2|)$$

$$\int \sin^n ax \cos ax\ dx = \frac{1}{a(n+1)}\sin^{n+1} ax + C \qquad (\text{for } n \neq -1)$$

$$\int \sin ax \cos^n ax\ dx = -\frac{1}{a(n+1)}\cos^{n+1} ax + C \qquad (\text{for } n \neq -1)$$

$$\int \sin^n ax \cos^m ax\ dx = -\frac{\sin^{n-1} ax \cos^{m+1} ax}{a(n+m)} + \frac{n-1}{n+m}\int \sin^{n-2} ax \cos^m ax\ dx \qquad (\text{f}$$

also:

$$\int \sin^n ax \cos^m ax\ dx = \frac{\sin^{n+1} ax \cos^{m-1} ax}{a(n+m)} + \frac{m-1}{n+m}\int \sin^n ax \cos^{m-2} ax\ dx \qquad (\text{for}$$

$$\int \frac{dx}{\sin ax \cos ax} = \frac{1}{a}\ln|\tan ax| + C$$

$$\int \frac{dx}{\sin ax \cos^n ax} = \frac{1}{a(n-1)\cos^{n-1} ax} + \int \frac{dx}{\sin ax \cos^{n-2} ax} \qquad (\text{for } n \neq 1)$$

$$\int \frac{dx}{\sin^n ax \cos ax} = -\frac{1}{a(n-1)\sin^{n-1} ax} + \int \frac{dx}{\sin^{n-2} ax \cos ax} \qquad (\text{for } n \neq 1)$$

$$\int \frac{\sin ax\ dx}{\cos^n ax} = \frac{1}{a(n-1)\cos^{n-1} ax} + C \qquad (\text{for } n \neq 1)$$

$$\int \frac{\sin^2 ax\ dx}{\cos ax} = -\frac{1}{a}\sin ax + \frac{1}{a}\ln\left|\tan\left(\frac{\pi}{4}+\frac{ax}{2}\right)\right| + C$$

$$\int \frac{\sin^2 ax\ dx}{\cos^n ax} = \frac{\sin ax}{a(n-1)\cos^{n-1} ax} - \frac{1}{n-1}\int \frac{dx}{\cos^{n-2} ax} \qquad (\text{for } n \neq 1)$$

$$\int \frac{\sin^n ax\ dx}{\cos ax} = -\frac{\sin^{n-1} ax}{a(n-1)} + \int \frac{\sin^{n-2} ax\ dx}{\cos ax} \qquad (\text{for } n \neq 1)$$

$$\int \frac{\sin^n ax\ dx}{\cos^m ax} = \frac{\sin^{n+1} ax}{a(m-1)\cos^{m-1} ax} - \frac{n-m+2}{m-1}\int \frac{\sin^n ax\ dx}{\cos^{m-2} ax} \qquad (\text{for } m \neq 1)$$

also:

$$\int \frac{\sin^n ax\ dx}{\cos^m ax} = -\frac{\sin^{n-1} ax}{a(n-m)\cos^{m-1} ax} + \frac{n-1}{n-m}\int \frac{\sin^{n-2} ax\ dx}{\cos^m ax} \qquad (\text{for } m \neq n)$$

also:

$$\int \frac{\sin^n ax\ dx}{\cos^m ax} = \frac{\sin^{n-1} ax}{a(m-1)\cos^{m-1} ax} - \frac{n-1}{m-1}\int \frac{\sin^{n-2} ax\ dx}{\cos^{m-2} ax} \qquad (\text{for } m \neq 1)$$

$$\int \frac{\cos ax\ dx}{\sin^n ax} = -\frac{1}{a(n-1)\sin^{n-1} ax} + C \qquad (\text{for } n \neq 1)$$

$$\int \frac{\cos^2 ax\ dx}{\sin ax} = \frac{1}{a}\left(\cos ax + \ln\left|\tan\frac{ax}{2}\right|\right) + C$$
$$\int \frac{\cos^2 ax\ dx}{\sin^n ax} = -\frac{1}{n-1}\left(\frac{\cos ax}{a\sin^{n-1} ax)} + \int \frac{dx}{\sin^{n-2} ax}\right) \qquad (\text{for } n \neq 1)$$
$$\int \frac{\cos^n ax\ dx}{\sin^m ax} = -\frac{\cos^{n+1} ax}{a(m-1)\sin^{m-1} ax} - \frac{n-m-2}{m-1}\int \frac{\cos^n ax\ dx}{\sin^{m-2} ax} \qquad (\text{for } m \neq 1)$$
also:
$$\int \frac{\cos^n ax\ dx}{\sin^m ax} = \frac{\cos^{n-1} ax}{a(n-m)\sin^{m-1} ax} + \frac{n-1}{n-m}\int \frac{\cos^{n-2} ax\ dx}{\sin^m ax} \qquad (\text{for } m \neq n)$$
also:
$$\int \frac{\cos^n ax\ dx}{\sin^m ax} = -\frac{\cos^{n-1} ax}{a(m-1)\sin^{m-1} ax} - \frac{n-1}{m-1}\int \frac{\cos^{n-2} ax\ dx}{\sin^{m-2} ax} \qquad (\text{for } m \neq 1)$$

13.9 TRIGONOMETRIC FUNCTIONS (SINE & TANGENT):

$$\int \sin ax \tan ax\ dx = \frac{1}{a}(\ln|\sec ax + \tan ax| - \sin ax) + C$$
$$\int \frac{\tan^n ax\ dx}{\sin^2 ax} = \frac{1}{a(n-1)}\tan^{n-1}(ax) + C \qquad (\text{for } n \neq 1)$$

13.10 TRIGONOMETRIC FUNCTIONS (COSINE & TANGENT):

$$\int \frac{\tan^n ax\ dx}{\cos^2 ax} = \frac{1}{a(n+1)}\tan^{n+1} ax + C \qquad (\text{for } n \neq -1)$$

13.11 TRIGONOMETRIC FUNCTIONS (SINE & COTANGENT):

$$\int \frac{\cot^n ax\ dx}{\sin^2 ax} = \frac{1}{a(n+1)}\cot^{n+1} ax + C \qquad (\text{for } n \neq -1)$$

13.12 TRIGONOMETRIC FUNCTIONS (COSINE & COTANGENT):

$$\int \frac{\cot^n ax\ dx}{\cos^2 ax} = \frac{1}{a(1-n)}\tan^{1-n} ax + C \qquad (\text{for } n \neq 1)$$

13.13 TRIGONOMETRIC FUNCTIONS (ARCSINE):

$$\int \arcsin x\ dx = x\arcsin x + \sqrt{1-x^2} + C$$
$$\int \arcsin\frac{x}{a}\ dx = x\arcsin\frac{x}{a} + \sqrt{a^2-x^2} + C$$
$$\int x\arcsin\frac{x}{a}\ dx = \left(\frac{x^2}{2} - \frac{a^2}{4}\right)\arcsin\frac{x}{a} + \frac{x}{4}\sqrt{a^2-x^2} + C$$

$$\int x^2 \arcsin\frac{x}{a}\ dx = \frac{x^3}{3}\arcsin\frac{x}{a} + \frac{x^2+2a^2}{9}\sqrt{a^2-x^2} + C$$
$$\int x^n \arcsin x\, dx = \frac{1}{n+1}\left(x^{n+1}\arcsin x + \frac{x^n\sqrt{1-x^2} - nx^{n-1}\arcsin x}{n-1} + n\int x^{n-2}\arcsin x\ dx\right)$$
$$\int \cos^n x \arcsin x\, dx = \left(x^{n^2+1}\arccos x + \frac{x^n\sqrt{1-x^4} - nx^{n^2-1}\arccos x}{n^2-1} + n\int x^{n^2-2}\arccos x\ dx\right)$$

13.14 TRIGONOMETRIC FUNCTIONS (ARCCOSINE):

$$\int \arccos x\, dx = x\arccos x - \sqrt{1-x^2} + C$$
$$\int \arccos\frac{x}{a}\ dx = x\arccos\frac{x}{a} - \sqrt{a^2-x^2} + C$$
$$\int x\arccos\frac{x}{a}\ dx = \left(\frac{x^2}{2} - \frac{a^2}{4}\right)\arccos\frac{x}{a} - \frac{x}{4}\sqrt{a^2-x^2} + C$$
$$\int x^2\arccos\frac{x}{a}\ dx = \frac{x^3}{3}\arccos\frac{x}{a} - \frac{x^2+2a^2}{9}\sqrt{a^2-x^2} + C$$

13.15 TRIGONOMETRIC FUNCTIONS (ARCTANGENT):

$$\int \arctan x\, dx = x\arctan x - \frac{1}{2}\ln(1+x^2) + C$$
$$\int \arctan(\frac{x}{a})dx = x\arctan(\frac{x}{a}) - \frac{a}{2}\ln(1+\frac{x^2}{a^2}) + C$$
$$\int x\arctan(\frac{x}{a})dx = \frac{(a^2+x^2)\arctan(\frac{x}{a}) - ax}{2} + C$$
$$\int x^2\arctan(\frac{x}{a})dx = \frac{x^3}{3}\arctan(\frac{x}{a}) - \frac{ax^2}{6} + \frac{a^3}{6}\ln(a^2+x^2) + C$$
$$\int x^n\arctan(\frac{x}{a})dx = \frac{x^{n+1}}{n+1}\arctan(\frac{x}{a}) - \frac{a}{n+1}\int\frac{x^{n+1}}{a^2+x^2}\ dx, \quad n \neq -1$$

13.16 TRIGONOMETRIC FUNCTIONS (ARCCOSECANT):

$$\int \operatorname{arccsc} x\, dx = x\operatorname{arccsc} x + \ln\left|x + x\sqrt{\frac{x^2-1}{x^2}}\right| + C$$
$$\int \operatorname{arccsc}\frac{x}{a}\ dx = x\operatorname{arccsc}\frac{x}{a} + a\ln(\frac{x}{a}(\sqrt{1-\frac{a^2}{x^2}}+1)) + C$$
$$\int x\operatorname{arccsc}\frac{x}{a}\ dx = \frac{x^2}{2}\operatorname{arccsc}\frac{x}{a} + \frac{ax}{2}\sqrt{1-\frac{a^2}{x^2}} + C$$

13.17 TRIGONOMETRIC FUNCTIONS (ARCSECANT):

$$\int \operatorname{arcsec} x \, dx = x \operatorname{arcsec} x - \ln\left|x + x\sqrt{\frac{x^2-1}{x^2}}\right| + C$$

$$\int \operatorname{arcsec} \frac{x}{a} \, dx = x \operatorname{arcsec} \frac{x}{a} + \frac{x}{a|x|} \ln\left|x \pm \sqrt{x^2-1}\right| + C$$

$$\int x \operatorname{arcsec} x \, dx = \frac{1}{2}\left(x^2 \operatorname{arcsec} x - \sqrt{x^2-1}\right) + C$$

$$\int x^n \operatorname{arcsec} x \, dx = \frac{1}{n+1}\left(x^{n+1} \operatorname{arcsec} x - \frac{1}{n}\left[x^{n-1}\sqrt{x^2-1} + [1-n]\left(x^{n-1} \operatorname{arcsec} x + (1-n)\int x^{n-2} \operatorname{arcsec} x \, dx\right)\right]\right)$$

13.18 TRIGONOMETRIC FUNCTIONS (ARCCOTANGENT):

$$\int \operatorname{arccot} x \, dx = x \operatorname{arccot} x + \frac{1}{2}\ln(1+x^2) + C$$

$$\int \operatorname{arccot} \frac{x}{a} \, dx = x \operatorname{arccot} \frac{x}{a} + \frac{a}{2}\ln(a^2+x^2) + C$$

$$\int x \operatorname{arccot} \frac{x}{a} \, dx = \frac{a^2+x^2}{2} \operatorname{arccot} \frac{x}{a} + \frac{ax}{2} + C$$

$$\int x^2 \operatorname{arccot} \frac{x}{a} \, dx = \frac{x^3}{3} \operatorname{arccot} \frac{x}{a} + \frac{ax^2}{6} - \frac{a^3}{6}\ln(a^2+x^2) + C$$

$$\int x^n \operatorname{arccot} \frac{x}{a} \, dx = \frac{x^{n+1}}{n+1} \operatorname{arccot} \frac{x}{a} + \frac{a}{n+1}\int \frac{x^{n+1}}{a^2+x^2} \, dx, \quad n \neq -1$$

13.19 EXPONETIAL FUNCTIONS

$$\int e^x \, \mathrm{d}x = e^x$$

$$\int e^{cx} \, \mathrm{d}x = \frac{1}{c} e^{cx}$$

$$\int a^{cx} \, \mathrm{d}x = \frac{1}{c \cdot \ln a} a^{cx} \text{ for } a > 0, \; a \neq 1$$

$$\int x e^{cx} \, \mathrm{d}x = \frac{e^{cx}}{c^2}(cx-1)$$

$$\int x^2 e^{cx} \, \mathrm{d}x = e^{cx}\left(\frac{x^2}{c} - \frac{2x}{c^2} + \frac{2}{c^3}\right)$$

$$\int x^n e^{cx} \, \mathrm{d}x = \frac{1}{c} x^n e^{cx} - \frac{n}{c}\int x^{n-1} e^{cx} \mathrm{d}x$$

$$\int \frac{e^{cx}}{x} \, \mathrm{d}x = \ln|x| + \sum_{n=1}^{\infty} \frac{(cx)^n}{n \cdot n!}$$

$$\int \frac{e^{cx}}{x^n}\,\mathrm{d}x = \frac{1}{n-1}\left(-\frac{e^{cx}}{x^{n-1}} + c\int \frac{e^{cx}}{x^{n-1}}\,\mathrm{d}x\right) \qquad (\text{for } n \neq 1)$$

$$\int e^{cx}\ln x\,\mathrm{d}x = \frac{1}{c}e^{cx}\ln|x| - \operatorname{Ei}(cx)$$

$$\int e^{cx}\sin bx\,\mathrm{d}x = \frac{e^{cx}}{c^2+b^2}(c\sin bx - b\cos bx)$$

$$\int e^{cx}\cos bx\,\mathrm{d}x = \frac{e^{cx}}{c^2+b^2}(c\cos bx + b\sin bx)$$

$$\int e^{cx}\sin^n x\,\mathrm{d}x = \frac{e^{cx}\sin^{n-1}x}{c^2+n^2}(c\sin x - n\cos x) + \frac{n(n-1)}{c^2+n^2}\int e^{cx}\sin^{n-2}x\,\mathrm{d}x$$

$$\int e^{cx}\cos^n x\,\mathrm{d}x = \frac{e^{cx}\cos^{n-1}x}{c^2+n^2}(c\cos x + n\sin x) + \frac{n(n-1)}{c^2+n^2}\int e^{cx}\cos^{n-2}x\,\mathrm{d}x$$

$$\int xe^{cx^2}\,\mathrm{d}x = \frac{1}{2c}\,e^{cx^2}$$

$$\int e^{-cx^2}\,\mathrm{d}x = \sqrt{\frac{\pi}{4c}}\operatorname{erf}(\sqrt{c}x)$$ (erf is the Error function)

$$\int xe^{-cx^2}\,\mathrm{d}x = -\frac{1}{2c}e^{-cx^2}$$

$$\int \frac{1}{\sigma\sqrt{2\pi}}\,e^{-(x-\mu)^2/2\sigma^2}\,\mathrm{d}x = \frac{1}{2}\left(1 + \operatorname{erf}\frac{x-\mu}{\sigma\sqrt{2}}\right)$$

$$\int e^{x^2}\,\mathrm{d}x = e^{x^2}\left(\sum_{j=0}^{n-1} c_{2j}\,\frac{1}{x^{2j+1}}\right) + (2n-1)c_{2n-2}\int \frac{e^{x^2}}{x^{2n}}\,\mathrm{d}x \quad \text{valid for } n > 0,$$

where $c_{2j} = \frac{1\cdot 3\cdot 5\cdots(2j-1)}{2^{j+1}} = \frac{(2j)\,!}{j!\,2^{2j+1}}$.

$$\int \underbrace{x^{x^{\cdot^{\cdot^{x}}}}}_{m}\,dx = \sum_{n=0}^{m} \frac{(-1)^n(n+1)^{n-1}}{n!}\Gamma(n+1,-\ln x) + \sum_{n=m+1}^{\infty}(-1)^n a_{mn}\Gamma(n+1,-\ln x) \qquad (\text{for } x > 0)$$

where $a_{mn} = \begin{cases} 1 & \text{if } n = 0, \\ \frac{1}{n!} & \text{if } m = 1, \\ \frac{1}{n}\sum_{j=1}^{n} j a_{m,n-j} a_{m-1,j-1} & \text{otherwise} \end{cases}$

$$\int \frac{1}{ae^{\lambda x}+b}\,\mathrm{d}x = \frac{x}{b} - \frac{1}{b\lambda}\ln\left(ae^{\lambda x}+b\right)$$ when $b \neq 0$, $\lambda \neq 0$ and $ae^{\lambda x} + b > 0$.

$$\int_0^1 e^{x\cdot\ln a + (1-x)\cdot\ln b}\,\mathrm{d}x = \int_0^1 \left(\frac{a}{b}\right)^x \cdot b\,\mathrm{d}x = \int_0^1 a^x \cdot b^{1-x}\,\mathrm{d}x = \frac{a-b}{\ln a - \ln b}$$

for $a > 0,\ b > 0,\ a \neq b$, which is the logarithmic mean

$$\int_0^{\infty} e^{-ax}\,\mathrm{d}x = \frac{1}{a}$$

$$\int_0^\infty e^{-ax^2}\,\mathrm{d}x = \frac{1}{2}\sqrt{\frac{\pi}{a}} \quad (a>0)$$
$$\int_{-\infty}^\infty e^{-ax^2}\,\mathrm{d}x = \sqrt{\frac{\pi}{a}} \quad (a>0)$$
$$\int_{-\infty}^\infty e^{-ax^2}e^{-2bx}\,\mathrm{d}x = \sqrt{\frac{\pi}{a}}e^{\frac{b^2}{a}} \quad (a>0)\int_{-\infty}^\infty xe^{-a(x-b)^2}\,\mathrm{d}x = b\sqrt{\frac{\pi}{a}}$$
$$\int_{-\infty}^\infty x^2e^{-ax^2}\,\mathrm{d}x = \frac{1}{2}\sqrt{\frac{\pi}{a^3}} \quad (a>0)$$
$$\int_0^\infty x^ne^{-ax^2}\,\mathrm{d}x = \begin{cases} \frac{1}{2}\Gamma\left(\frac{n+1}{2}\right)/a^{\frac{n+1}{2}} & (n>-1, a>0) \\ \frac{(2k-1)!!}{2^{k+1}a^k}\sqrt{\frac{\pi}{a}} & (n=2k, k \text{ integer}, a>0) \\ \frac{k!}{2a^{k+1}} & (n=2k+1, k \text{ integer}, a>0) \end{cases}$$
(!! is the double factorial)
$$\int_0^\infty x^ne^{-ax}\,\mathrm{d}x = \begin{cases} \frac{\Gamma(n+1)}{a^{n+1}} & (n>-1, a>0) \\ \frac{n!}{a^{n+1}} & (n=0,1,2,\ldots, a>0) \end{cases}$$
$$\int_0^\infty e^{-ax}\sin bx\,\mathrm{d}x = \frac{b}{a^2+b^2} \quad (a>0)$$
$$\int_0^\infty e^{-ax}\cos bx\,\mathrm{d}x = \frac{a}{a^2+b^2} \quad (a>0)$$
$$\int_0^\infty xe^{-ax}\sin bx\,\mathrm{d}x = \frac{2ab}{(a^2+b^2)^2} \quad (a>0)$$
$$\int_0^\infty xe^{-ax}\cos bx\,\mathrm{d}x = \frac{a^2-b^2}{(a^2+b^2)^2} \quad (a>0)$$
$$\int_0^{2\pi} e^{x\cos\theta}d\theta = 2\pi I_0(x)$$
(I_0 is the modified Bessel function of the first kind)
$$\int_0^{2\pi} e^{x\cos\theta+y\sin\theta}d\theta = 2\pi I_0\left(\sqrt{x^2+y^2}\right)$$

13.20 LOGARITHMIC FUNCTIONS

$$\int \ln(x)\,dx = x\ln(x) - x + C,$$
$$\int \ln ax\;dx = x\ln ax - x$$
$$\int \log_a x\,dx = x(\log_a x - \log_a e) + C$$
$$\int \ln(ax+b)\;dx = \frac{(ax+b)\ln(ax+b)-(ax)}{a}$$
$$\int (\ln x)^2\;dx = x(\ln x)^2 - 2x\ln x + 2x$$
$$\int (\ln x)^n\;dx = x\sum_{k=0}^{n}(-1)^{n-k}\frac{n!}{k!}(\ln x)^k$$

$$\int \frac{dx}{\ln x} = \ln|\ln x| + \ln x + \sum_{k=2}^{\infty} \frac{(\ln x)^k}{k \cdot k!}$$

$$\int \frac{dx}{(\ln x)^n} = -\frac{x}{(n-1)(\ln x)^{n-1}} + \frac{1}{n-1} \int \frac{dx}{(\ln x)^{n-1}} \qquad (\text{for } n \neq 1)$$

$$\int x^m \ln x \, dx = x^{m+1} \left(\frac{\ln x}{m+1} - \frac{1}{(m+1)^2} \right) \qquad (\text{for } m \neq -1)$$

$$\int x^m (\ln x)^n \, dx = \frac{x^{m+1} (\ln x)^n}{m+1} - \frac{n}{m+1} \int x^m (\ln x)^{n-1} dx \qquad (\text{for } m \neq -1)$$

$$\int \frac{(\ln x)^n \, dx}{x} = \frac{(\ln x)^{n+1}}{n+1} \qquad (\text{for } n \neq -1)$$

$$\int \frac{\ln x^n \, dx}{x} = \frac{(\ln x^n)^2}{2n} \qquad (\text{for } n \neq 0)$$

$$\int \frac{\ln x \, dx}{x^m} = -\frac{\ln x}{(m-1)x^{m-1}} - \frac{1}{(m-1)^2 x^{m-1}} \qquad (\text{for } m \neq 1)$$

$$\int \frac{(\ln x)^n \, dx}{x^m} = -\frac{(\ln x)^n}{(m-1)x^{m-1}} + \frac{n}{m-1} \int \frac{(\ln x)^{n-1} dx}{x^m} \qquad (\text{for } m \neq 1)$$

$$\int \frac{x^m \, dx}{(\ln x)^n} = -\frac{x^{m+1}}{(n-1)(\ln x)^{n-1}} + \frac{m+1}{n-1} \int \frac{x^m dx}{(\ln x)^{n-1}} \qquad (\text{for } n \neq 1)$$

$$\int \frac{dx}{x \ln x} = \ln|\ln x|$$

$$\int \frac{dx}{x^n \ln x} = \ln|\ln x| + \sum_{k=1}^{\infty} (-1)^k \frac{(n-1)^k (\ln x)^k}{k \cdot k!}$$

$$\int \frac{dx}{x (\ln x)^n} = -\frac{1}{(n-1)(\ln x)^{n-1}} \qquad (\text{for } n \neq 1)$$

$$\int \ln(x^2 + a^2) \, dx = x \ln(x^2 + a^2) - 2x + 2a \tan^{-1} \frac{x}{a}$$

$$\int \frac{x}{x^2 + a^2} \ln(x^2 + a^2) \, dx = \frac{1}{4} \ln^2(x^2 + a^2)$$

$$\int \sin(\ln x) \, dx = \frac{x}{2} (\sin(\ln x) - \cos(\ln x))$$

$$\int \cos(\ln x) \, dx = \frac{x}{2} (\sin(\ln x) + \cos(\ln x))$$

$$\int e^x \left(x \ln x - x - \frac{1}{x} \right) \, dx = e^x (x \ln x - x - \ln x)$$

$$\int \frac{1}{e^x} \left(\frac{1}{x} - \ln x \right) \, dx = \frac{\ln x}{e^x}$$

$$\int (\ln x)^x \, dx = (\ln x)^{x-1} + (\ln(\ln x))(\ln x)^x$$

$$\int e^x \left(\frac{1}{\ln x} - \frac{1}{x \ln^2 x} \right) \, dx = \frac{e^x}{\ln x}$$

13.21 HYPERBOLIC FUNCTIONS

$$\int \sinh ax\,dx = \frac{1}{a}\cosh ax + C$$
$$\int \cosh ax\,dx = \frac{1}{a}\sinh ax + C$$
$$\int \sinh^2 ax\,dx = \frac{1}{4a}\sinh 2ax - \frac{x}{2} + C$$
$$\int \cosh^2 ax\,dx = \frac{1}{4a}\sinh 2ax + \frac{x}{2} + C$$
$$\int \tanh^2 ax\,dx = x - \frac{\tanh ax}{a} + C$$
$$\int \sinh^n ax\,dx = \frac{1}{an}\sinh^{n-1} ax \cosh ax - \frac{n-1}{n}\int \sinh^{n-2} ax\,dx \qquad \text{(for } n > 0\text{)}$$
$$\int \sinh^n ax\,dx = \frac{1}{a(n+1)}\sinh^{n+1} ax \cosh ax - \frac{n+2}{n+1}\int \sinh^{n+2} ax\,dx \qquad \text{(for } n < 0,\ n \neq -1\text{)}$$
$$\int \cosh^n ax\,dx = \frac{1}{an}\sinh ax \cosh^{n-1} ax + \frac{n-1}{n}\int \cosh^{n-2} ax\,dx \qquad \text{(for } n > 0\text{)}$$
$$\int \cosh^n ax\,dx = -\frac{1}{a(n+1)}\sinh ax \cosh^{n+1} ax - \frac{n+2}{n+1}\int \cosh^{n+2} ax\,dx \qquad \text{(for } n < 0,\ n \neq -1\text{)}$$
$$\int \frac{dx}{\sinh ax} = \frac{1}{a}\ln\left|\tanh\frac{ax}{2}\right| + C$$
$$\int \frac{dx}{\sinh ax} = \frac{1}{a}\ln\left|\frac{\cosh ax - 1}{\sinh ax}\right| + C$$
$$\int \frac{dx}{\sinh ax} = \frac{1}{a}\ln\left|\frac{\sinh ax}{\cosh ax + 1}\right| + C$$
$$\int \frac{dx}{\sinh ax} = \frac{1}{a}\ln\left|\frac{\cosh ax - 1}{\cosh ax + 1}\right| + C$$
$$\int \frac{dx}{\cosh ax} = \frac{2}{a}\arctan e^{ax} + C$$
$$\int \frac{dx}{\sinh^n ax} = -\frac{\cosh ax}{a(n-1)\sinh^{n-1} ax} - \frac{n-2}{n-1}\int \frac{dx}{\sinh^{n-2} ax} \qquad \text{(for } n \neq 1\text{)}$$
$$\int \frac{dx}{\cosh^n ax} = \frac{\sinh ax}{a(n-1)\cosh^{n-1} ax} + \frac{n-2}{n-1}\int \frac{dx}{\cosh^{n-2} ax} \qquad \text{(for } n \neq 1\text{)}$$
$$\int \frac{\cosh^n ax}{\sinh^m ax}dx = \frac{\cosh^{n-1} ax}{a(n-m)\sinh^{m-1} ax} + \frac{n-1}{n-m}\int \frac{\cosh^{n-2} ax}{\sinh^m ax}dx \qquad \text{(for } m \neq n\text{)}$$
$$\int \frac{\cosh^n ax}{\sinh^m ax}dx = -\frac{\cosh^{n+1} ax}{a(m-1)\sinh^{m-1} ax} + \frac{n-m+2}{m-1}\int \frac{\cosh^n ax}{\sinh^{m-2} ax}dx \qquad \text{(for } m \neq 1\text{)}$$
$$\int \frac{\cosh^n ax}{\sinh^m ax}dx = -\frac{\cosh^{n-1} ax}{a(m-1)\sinh^{m-1} ax} + \frac{n-1}{m-1}\int \frac{\cosh^{n-2} ax}{\sinh^{m-2} ax}dx \qquad \text{(for } m \neq 1\text{)}$$
$$\int \frac{\sinh^m ax}{\cosh^n ax}dx = \frac{\sinh^{m-1} ax}{a(m-n)\cosh^{n-1} ax} + \frac{m-1}{n-m}\int \frac{\sinh^{m-2} ax}{\cosh^n ax}dx \qquad \text{(for } m \neq n\text{)}$$

$$\int \frac{\sinh^m ax}{\cosh^n ax}dx = \frac{\sinh^{m+1} ax}{a(n-1)\cosh^{n-1} ax}+\frac{m-n+2}{n-1}\int \frac{\sinh^m ax}{\cosh^{n-2} ax}dx \qquad (\text{for } n \neq 1)$$

$$\int \frac{\sinh^m ax}{\cosh^n ax}dx = -\frac{\sinh^{m-1} ax}{a(n-1)\cosh^{n-1} ax}+\frac{m-1}{n-1}\int \frac{\sinh^{m-2} ax}{\cosh^{n-2} ax}dx \qquad (\text{for } n \neq 1)$$

$$\int x\sinh ax\,dx = \frac{1}{a}x\cosh ax - \frac{1}{a^2}\sinh ax + C$$

$$\int x\cosh ax\,dx = \frac{1}{a}x\sinh ax - \frac{1}{a^2}\cosh ax + C$$

$$\int x^2\cosh ax\,dx = -\frac{2x\cosh ax}{a^2} + \left(\frac{x^2}{a} + \frac{2}{a^3}\right)\sinh ax + C$$

$$\int \tanh ax\,dx = \frac{1}{a}\ln|\cosh ax| + C$$

$$\int \coth ax\,dx = \frac{1}{a}\ln|\sinh ax| + C$$

$$\int \tanh^n ax\,dx = -\frac{1}{a(n-1)}\tanh^{n-1} ax + \int \tanh^{n-2} ax\,dx \qquad (\text{for } n \neq 1)$$

$$\int \coth^n ax\,dx = -\frac{1}{a(n-1)}\coth^{n-1} ax + \int \coth^{n-2} ax\,dx \qquad (\text{for } n \neq 1)$$

$$\int \sinh ax\sinh bx\,dx = \frac{1}{a^2-b^2}(a\sinh bx\cosh ax - b\cosh bx\sinh ax)+C \qquad (\text{for } a^2 \neq b^2)$$

$$\int \cosh ax\cosh bx\,dx = \frac{1}{a^2-b^2}(a\sinh ax\cosh bx - b\sinh bx\cosh ax)+C \qquad (\text{for } a^2 \neq b^2)$$

$$\int \cosh ax\sinh bx\,dx = \frac{1}{a^2-b^2}(a\sinh ax\sinh bx - b\cosh ax\cosh bx)+C \qquad (\text{for } a^2 \neq b^2)$$

$$\int \sinh(ax+b)\sin(cx+d)\,dx = \frac{a}{a^2+c^2}\cosh(ax+b)\sin(cx+d) - \frac{c}{a^2+c^2}\sinh(ax+b)\cos(cx+d)+C$$

$$\int \sinh(ax+b)\cos(cx+d)\,dx = \frac{a}{a^2+c^2}\cosh(ax+b)\cos(cx+d) + \frac{c}{a^2+c^2}\sinh(ax+b)\sin(cx+d)+C$$

$$\int \cosh(ax+b)\sin(cx+d)\,dx = \frac{a}{a^2+c^2}\sinh(ax+b)\sin(cx+d) - \frac{c}{a^2+c^2}\cosh(ax+b)\cos(cx+d)+C$$

$$\int \cosh(ax+b)\cos(cx+d)\,dx = \frac{a}{a^2+c^2}\sinh(ax+b)\cos(cx+d) + \frac{c}{a^2+c^2}\cosh(ax+b)\sin(cx+d)+C$$

13.22 INVERSE HYPERBOLIC FUNCTIONS

$$\int \operatorname{arsinh}\frac{x}{a}\,dx = x\operatorname{arsinh}\frac{x}{a} - \sqrt{x^2+a^2} + C$$

$$\int \operatorname{arcosh}\frac{x}{a}\,dx = x\operatorname{arcosh}\frac{x}{a} - \sqrt{x^2-a^2} + C$$

$$\int \operatorname{artanh}\frac{x}{a}\,dx = x\operatorname{artanh}\frac{x}{a} + \frac{a}{2}\ln|a^2-x^2| + C \qquad (\text{for } |x| < |a|)$$

$$\int \operatorname{arcoth}\frac{x}{a}\,dx = x\operatorname{arcoth}\frac{x}{a} + \frac{a}{2}\ln|x^2-a^2| + C \qquad (\text{for } |x| > |a|)$$

$$\int \operatorname{arsech} \frac{x}{a}\,dx = x \operatorname{arsech} \frac{x}{a} - a \arctan \frac{x\sqrt{\frac{a-x}{a+x}}}{x-a} + C \qquad (\text{for } x \in (0,\,a))$$

$$\int \operatorname{arcsch} \frac{x}{a}\,dx = x \operatorname{arcsch} \frac{x}{a} + a \ln \frac{x+\sqrt{x^2+a^2}}{a} + C \qquad (\text{for } x \in (0,\,a))$$

13.23 ABSOLUTE VALUE FUNCTIONS

$$\int |(ax+b)^n|\;dx = \frac{(ax+b)^{n+2}}{a(n+1)\,|ax+b|} + C \;[\,n \text{ is odd, and } n \neq -1\,]$$

$$\int |\sin ax|\;dx = \frac{-1}{a}\,|\sin ax| \cot ax + C$$

$$\int |\cos ax|\;dx = \frac{1}{a}\,|\cos ax| \tan ax + C$$

$$\int |\tan ax|\;dx = \frac{\tan(ax)[-\ln|\cos ax|]}{a\,|\tan ax|} + C$$

$$\int |\csc ax|\;dx = \frac{-\ln|\csc ax + \cot ax| \sin ax}{a\,|\sin ax|} + C$$

$$\int |\sec ax|\;dx = \frac{\ln|\sec ax + \tan ax| \cos ax}{a\,|\cos ax|} + C$$

$$\int |\cot ax|\;dx = \frac{\tan(ax)[\ln|\sin ax|]}{a\,|\tan ax|} + C$$

13.24 SUMMARY TABLE

$\int \frac{f'(u)}{f(u)}\,du = \ln\lvert f(u)\rvert + C, \qquad \int (f(u))^n\, f'(u)\,du = \frac{(f(u))^{n+1}}{n+1} + C$
$\int \sin u\,du = -\cos u + C, \quad \int \cos u\,du = \sin u + C, \quad \int \sec^2 u\,du = \tan u + C$
$\int \tan u\,du = \ln\lvert \sec u\rvert + C, \qquad \int \tanh u\,du = \ln(\cosh u) + C$
$\int \cot u\,du = \ln\lvert \sin u\rvert + C, \qquad \int \coth u\,du = \ln\lvert \sinh u\rvert + C$
$\int \sec u\,du = \ln\lvert \sec u + \tan u\rvert + C, \quad \int \operatorname{sech} u\,du = \tan^{-1}\lvert \sinh u\rvert + C$
$\int \operatorname{cosec} u\,du = \ln\lvert \operatorname{cosec} u - \cot u\rvert + C, \quad \int \operatorname{cosech} u\,du = \ln\lvert \tanh \frac{u}{2}\rvert + C$
$\int \frac{du}{a^2-u^2} = \frac{1}{2a}\ln\left\lvert\frac{u+a}{u-a}\right\rvert + C, \qquad \int \frac{du}{u^2-a^2} = \frac{1}{2a}\ln\left\lvert\frac{u-a}{u+a}\right\rvert + C$
$\int \frac{du}{\sqrt{a^2-u^2}} = \sin^{-1}\left(\frac{u}{a}\right) + C, \qquad \int \frac{du}{a^2+u^2} = \frac{1}{a}\tan^{-1}\left(\frac{u}{a}\right) + C$
$\int \frac{du}{u\sqrt{u^2-a^2}} = \frac{1}{a}\sec^{-1}\left(\frac{u}{a}\right) + C$
$\int \frac{du}{\sqrt{u^2+a^2}} = \sinh^{-1}\left(\frac{u}{a}\right) + C = \ln\lvert u + \sqrt{u^2+a^2}\rvert + C$
$\int \frac{du}{\sqrt{u^2-a^2}} = \cosh^{-1}\left(\frac{u}{a}\right) + C = \ln\lvert u + \sqrt{u^2-a^2}\rvert + C$
$\int \frac{du}{u\sqrt{a^2-u^2}} = -\frac{1}{a}\operatorname{sech}^{-1}\left(\frac{u}{a}\right) + C = -\frac{1}{a}\ln\left\lvert\frac{a+\sqrt{a^2-u^2}}{u}\right\rvert + C$

13.25 SQUARE ROOT PROOFS

$$\int \sqrt{a^2 + x^2}\, dx$$

Let $x = a\tan\theta \therefore dx = a\sec^2\theta d\theta \to \tan\theta = \frac{x}{a}$

$$= \int \sqrt{a^2 + (a\tan\theta)^2} \times a\sec^2\theta d\theta$$

$$= \int \sqrt{a^2 + a^2\tan^2\theta} \times a\sec^2\theta d\theta$$

$$= \int \sqrt{a^2 + a^2(\sec^2\theta - 1)} \times a\sec^2\theta d\theta$$

$$= \int \sqrt{a^2 + a^2\sec^2\theta - a^2} \times a\sec^2\theta d\theta$$

$$= \int \sqrt{a^2\sec^2\theta} \times a\sec^2\theta d\theta$$

$$= \int a\sec\theta \times a\sec^2\theta d\theta$$

$$= \int a^2\sec^3\theta d\theta$$

$$= a^2 \int \sec\theta \times \sec^2\theta d\theta$$

$u = \sec\theta, dv = \sec^2\theta d\theta$

$du = \sec\theta\tan\theta d\theta, v = \tan\theta$

$$\therefore a^2 \int \sec^3\theta d\theta = \sec\theta \times \tan\theta - \int \tan\theta \times \sec\theta\tan\theta d\theta$$

$$a^2 \int \sec^3\theta d\theta = \sec\theta\tan\theta - \int \tan^2\theta\sec\theta d\theta$$

$$a^2 \int \sec^3\theta d\theta = \sec\theta\tan\theta - \int (\sec^2\theta - 1)\sec\theta d\theta$$

$$a^2 \int \sec^3\theta d\theta = \sec\theta\tan\theta - \int \sec^3\theta - \sec\theta d\theta$$

$$a^2 \int \sec^3\theta d\theta = \sec\theta\tan\theta - \int \sec^3\theta d\theta + \int \sec\theta d\theta$$

$$2a^2 \int \sec^3\theta d\theta = \sec\theta\tan\theta + \int \sec\theta d\theta$$

$$\int \sec^3\theta d\theta = \frac{1}{2a^2}\left(\sec\theta\tan\theta + \int \sec\theta d\theta\right)$$

$$\int \sec^3\theta d\theta = \frac{1}{2a^2}\left(\sec\theta\tan\theta + \ln\left|\sec\theta + \tan\theta\right|\right) + C$$

$$\therefore \int \sqrt{a^2 + x^2}\, dx = \frac{1}{2a^2}\left(\frac{\sqrt{a^2 + x^2}}{a} \times \frac{s}{a} + \ln\left|\frac{\sqrt{a^2 + x^2}}{a} + \frac{s}{a}\right|\right) + C$$

$$\int \sqrt{a^2 - x^2}\,dx$$

Let $x = a\sin\theta \therefore dx = a\cos\theta d\theta \rightarrow \sin\theta = \frac{x}{a}$

$$= \int \sqrt{a^2 - (a\sin\theta)^2} \times a\cos\theta d\theta$$

$$= \int \sqrt{a^2 - a^2\sin^2\theta} \times a\cos\theta d\theta$$

$$= \int \sqrt{a^2 - a^2(1 - \cos^2\theta)} \times a\cos\theta d\theta$$

$$= \int \sqrt{a^2 - a^2 + a^2\cos^2\theta} \times a\cos\theta d\theta$$

$$= \int \sqrt{a^2\cos^2\theta} \times a\cos\theta d\theta$$

$$= \int a\cos\theta \times a\cos\theta d\theta$$

$$= \int a^2\cos^2\theta d\theta$$

$$= a^2 \int \cos^2\theta d\theta$$

$$= a^2 \int \frac{1 + \cos(2\theta)}{2} d\theta$$

$$= \frac{a^2}{2} \int 1 + \cos(2\theta) d\theta$$

$$= \frac{a^2}{2}\left[\theta + \frac{\sin(2\theta)}{2}\right] + C$$

$$= \frac{a^2}{2}\left[\theta + \frac{2\sin\theta\cos\theta}{2}\right] + C$$

$$= \frac{a^2}{2}[\theta + \sin\theta\cos\theta] + C$$

$$= \frac{a^2}{2}\left[\arcsin\left(\frac{x}{a}\right) + \frac{x}{a} \times \frac{\sqrt{a^2 - x^2}}{a}\right] + C$$

$$\int \sqrt{x^2 - a^2}\,dx$$

Let $x = a\sec\theta \therefore dx = a\sec\theta\tan\theta d\theta \rightarrow \sec\theta = \frac{x}{a}$

$$= \int \sqrt{(a\sec\theta)^2 - a^2} \times a\sec\theta\tan\theta d\theta$$

$$= \int \sqrt{a^2\sec^2\theta - a^2} \times a\sec\theta\tan\theta d\theta$$

$$= \int \sqrt{a^2(1+\tan^2\theta) - a^2} \times a\sec\theta\tan\theta d\theta$$

$$= \int \sqrt{a^2 + a^2\tan^2\theta - a^2} \times a\sec\theta\tan\theta d\theta$$

$$= \int \sqrt{a^2\tan^2\theta} \times a\sec\theta\tan\theta d\theta$$

$$= \int a\tan\theta \times a\sec\theta\tan\theta d\theta$$

$$= \int a^2\tan^2\theta\sec\theta d\theta$$

$$= a^2 \int \tan^2\theta\sec\theta d\theta$$

$$= a^2 \int (\sec^2\theta - 1)\sec\theta d\theta$$

$$= a^2 \int \sec^3\theta - \sec\theta d\theta$$

$$= a^2 \left(\int \sec^3\theta d\theta - \int \sec\theta d\theta \right)$$

$$= a^2 \left(\left(\frac{1}{2a^2} \left(\sec\theta\tan\theta + \ln|\sec\theta + \tan\theta| \right) \right) - \left(\ln|\sec\theta + \tan\theta| \right) \right) + C$$

$$= \frac{1}{2} \left(\sec\theta\tan\theta + \ln|\sec\theta + \tan\theta| \right) - a^2 \left(\ln|\sec\theta + \tan\theta| \right) + C$$

$$= \frac{1}{2} \left(\frac{x}{a} \times \frac{\sqrt{x^2 - a^2}}{a} + \ln\left| \frac{x}{a} + \frac{\sqrt{x^2 - a^2}}{a} \right| \right) - a^2 \left(\ln\left| \frac{x}{a} + \frac{\sqrt{x^2 - a^2}}{a} \right| \right) + C$$

$$= \frac{1}{2} \frac{x \times \sqrt{x^2 - a^2}}{a^2} + \frac{1}{2}\ln\left| \frac{x}{a} + \frac{\sqrt{x^2 - a^2}}{a} \right| - \ln\left| \frac{x}{a} + \frac{\sqrt{x^2 - a^2}}{a} \right| + C$$

$$= \frac{1}{2} \frac{x \times \sqrt{x^2 - a^2}}{a^2} + \left(\frac{1}{2} - a^2 \right) \ln\left| \frac{x}{a} + \frac{\sqrt{x^2 - a^2}}{a} \right| + C$$

13.26 CARTESIAN APPLICATIONS

Area under the curve: $A = \int_a^b f_{(x)}dx$

Volume: $V = \int_a^b A$

Volume about x axis: $V_x = \pi\int_a^b [y]^2 dx = \pi\int_a^b \left[f_{(x)} \right]^2 dx$

Volume about y axis: $V_y = \pi\int_c^d [x]^2 dy$

Surface Area about x axis: $SA = 2\pi\int_a^b f_{(x)}\sqrt{1+\left(f'_{(x)}\right)^2}\,dx$

Length wrt x-ordinates: $L = \int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$

Length wrt y-ordinates: $L = \int_c^d \sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy$ (Where the function is continually increasing)

Length parametrically: $L = \int_{t1}^{t2} \sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt$

Line Integral of a Scalar Field:

$$L = \int_C f(x,y,z)ds = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i, y_y, z_y)\Delta s_i$$

where $x = x(t), y = y(t), z = z(t), a \le t \le b$

$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2+\left(\frac{dz}{dt}\right)^2}\,dt$$

$$L = \int_C f(x,y,z)ds = \int_a^b f(x,y,z)\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2+\left(\frac{dz}{dt}\right)^2}\,dt$$

Line Integral of a Vector Field:

$$W = \int_C F(x,y,z)\bullet T(x,y,z)ds$$

where $x = x(t), y = y(t), z = z(t), a \le t \le b$

$$C = r(t) = x(t)\underline{i} + y(t)\underline{j} + z(t)\underline{k}$$

$$T(t) = \frac{r'(t)}{|r'(t)|}$$

$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2+\left(\frac{dz}{dt}\right)^2}\,dt = |r'(t)|dt$$

$$W = \int_C F(x,y,z)\bullet T(x,y,z)ds = \int_a^b\left(F(r(t)\bullet\frac{r'(t)}{|r'(t)|}\right)|r'(t)|dt = \int_a^b F(r(t)\bullet r'(t)dt$$

Area of a Surface:

$$A = \iint_S f(x,y,z)dS = \lim_{m,n\to\infty}\sum_{i=1}^{m}\sum_{j=1}^{n} f(P_{ij}^*)\Delta S_{ij}$$

$$A = \iint_S f(x,y,z)dS = \iint_D f(x,y,g(x,y))\sqrt{\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2+1}\,dA$$

Where D is the projection of the surface S on the (x,y) plane

13.27 HIGHER ORDER INTEGRATION

Properties of Double Integrals:

$$\iint_R (f(x,y) \pm g(x,y))dA = \iint_R f(x,y)dA \pm \iint_R g(x,y)dA$$

$$\iint_R kf(x,y)dA = k\iint_R f(x,y)dA$$

$$\iint_R f(x,y)dA \leq \iint_R g(x,y)dA \quad \text{If } f(x,y) \leq g(x,y) \text{ on } R$$

$$\iint_S f(x,y)dA \leq \iint_R f(x,y)dA \quad \text{If } f(x,y) \geq 0 \text{ on } R \text{ and } S \subset R$$

Volume using Double Integrals:

$$V = \iint_R f(x,y)dA$$

:

If f(x, y) is continuous on the rectangle (Fubini's Theorem)

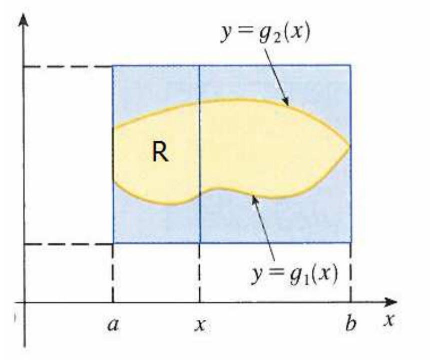
$R = \{(x,y) \mid a \leq x \leq b, c \leq y \leq d\}$, then

$$V = \iint_R f(x,y)dA = \int_a^b \int_c^d f(x,y)dxdy$$

If f(x, y) is continuous on the region

$$R = \{(x,y) \mid a \leq x \leq b, g_1(x) \leq y \leq g_2(x)\}$$

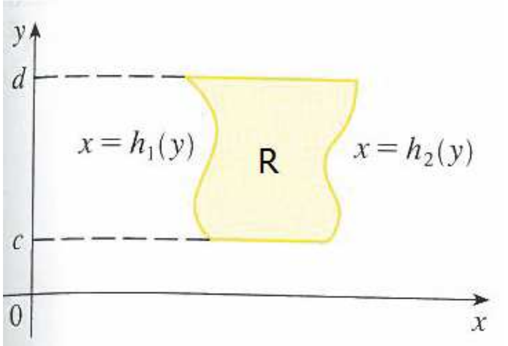
$$V = \iint_R f(x,y)dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)dydx$$



If f(x, y) is continuous on the region

$$R = \{(x,y) \mid g_1(y) \leq x \leq g_2(y), c \leq y \leq d\}$$

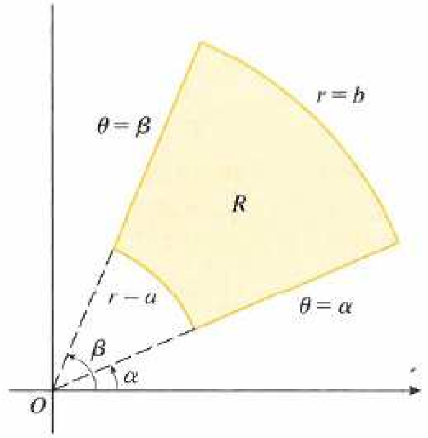
$$V = \iint_R f(x,y)A = \int_a^b \int_{h_1(y)}^{h_2(y)} f(x,y)dxdx$$



If a function is continuous on a polar region

$R = \{(r,\theta) \mid a \le r \le b, \alpha \le \theta \le \beta\}$

$$V = \iint_R f(x,y)A = \int_{\alpha}^{\beta}\int_{a}^{b} f(r\cos\theta, r\sin\theta) r\,dr\,d\theta$$



Volume using Triple Integrals:

$$V = \iiint_B f(x,y,z)dV$$

If f(x,y,z) is continuous on the rectangular box (Fubini's Theorem)

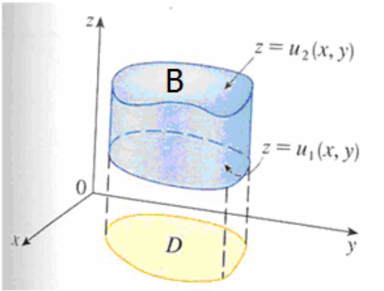
$B = \{(x,y,z) \mid a \le x \le b, c \le y \le d, r \le z \le s\}$, then

$$V = \iiint_B f(x,y,z)dV = \int_{r}^{s}\int_{c}^{d}\int_{a}^{b} f(x,y,z)\,dx\,dy\,dz$$

If f(x,y,z) is represented as a projection

$B = \{(x,y,z) \mid (x,y) \in D, u_1(x,y) \le z \le u_2(x,y)\}$, then

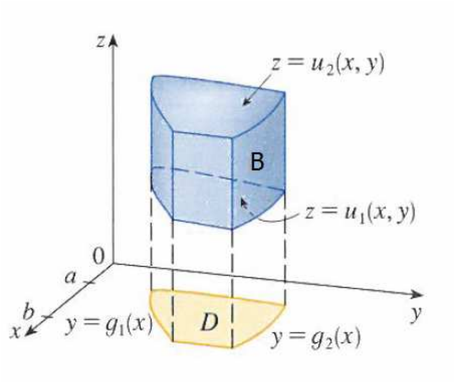
$$V = \iiint_B f(x,y,z)dV = \iint_D \left(\int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)dz \right) dA$$



If f(x,y,z) is continuous on the region

$$B = \{(x, y, z) \mid a \le x \le b, g_1(x) \le y \le g_2(x), u_1(x, y) \le z \le u_2(x, y)\}$$

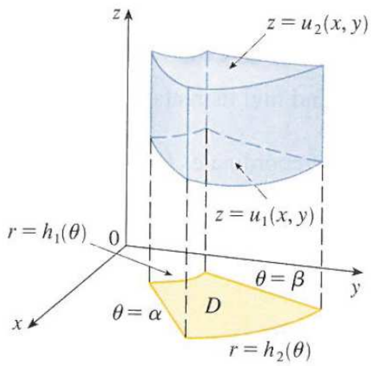
$$V = \iiint_B f(x, y, z)dV = \int_a^b \int_{g_1(x)}^{g_2(x)} \int_{u_1(x,y)}^{u_2(x,y)} f(x, y, z)dzdydx$$



If a function is continuous within the cylindrical coordinate system

$$B = \{(r, \theta, z) \mid \alpha \le \theta \le \beta, h_1(\theta) \le r \le h_2(\theta), u_1(x, y) \le z \le u_2(x, y)\}$$

$$V = \iiint_B f(x, y, z)dV = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} \int_{u_1(r\cos\theta, r\sin\theta)}^{u_2(r\cos\theta, r\sin\theta)} f(r\cos\theta, r\sin\theta, z)rdzdrd\theta$$

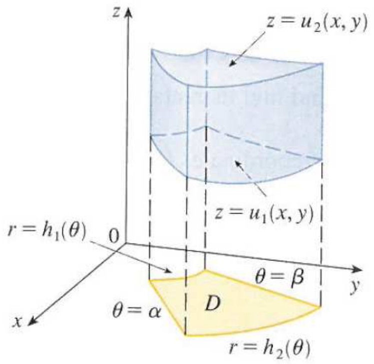


If a function is continuous within the spherical coordinate system

$$B = \{(\rho,\phi,\theta) \mid \theta_1 \le \theta \le \theta_2, h_1(\theta) \le \phi \le h_2(\theta), g_1(\phi,\theta) \le \rho \le g_2(\phi,\theta)\}$$

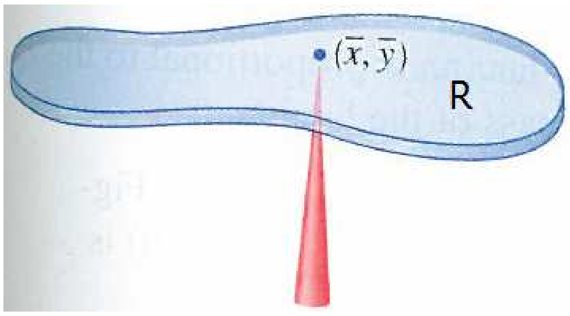
$$f(x,y,z) = f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi) = f_0(\rho,\phi,\theta)$$

$$V = \iiint_B f(x,y,z)dV = \int_{\theta_1}^{\theta_2}\int_{h_1(\theta)}^{h_2(\theta)}\int_{g_1(\phi,\theta)}^{g_2(\phi,\theta))} f_0(\rho,\phi,\theta)\rho^2 \sin\phi d\rho d\phi d\theta$$



Centre of Mass:

Of a laminate:



$$m = \iint_R \rho(x,y)dA$$

$$\bar{x} = \frac{1}{m}\iint_R x \times \rho(x,y)dA$$

$$\bar{y} = \frac{1}{m}\iint_R y \times \rho(x,y)dA$$

Of a general solid:

$$m = \iiint_B \rho(x,y,z)dV$$

$$\bar{x} = \frac{1}{m}\iiint_B x \times \rho(x,y,z)dV$$

$$\bar{y} = \frac{1}{m}\iiint_B y \times \rho(x,y,z)dV$$

$$\bar{z} = \frac{1}{m}\iiint_B z \times \rho(x,y,z)dV$$

13.28 WORKING IN DIFFERENT COORDINATE SYSTEMS:

Cartesian: $dA = dxdy$

Polar: $dA = rdrd\theta$

Cylindrical: $dV = rdzdrd\theta$

Spherical: $dV = \rho^2 \sin\phi d\rho d\phi d\theta$

Cartesian to Polar:

$$r^2 = x^2 + y^2$$
$$\tan\theta = \frac{y}{x} \quad \textit{(2D)}$$

Polar to Cartesian:

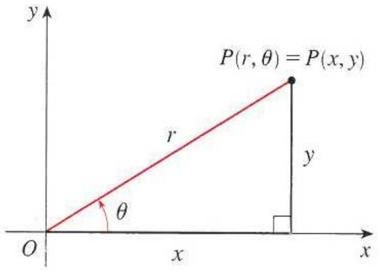
$$x = r\cos\theta$$
$$y = r\sin\theta \quad \textit{(2D)}$$

Cartesian to Cylindrical:

$$r^2 = x^2 + y^2$$
$$\tan\theta = \frac{y}{x} \quad \textit{(3D)}$$
$$z = z$$

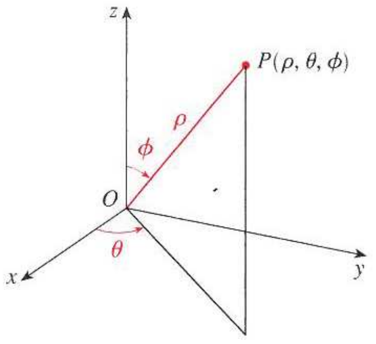
Cylindrical to Cartesian:

$$x = r\cos\theta$$
$$y = r\sin\theta \quad \textit{(3D)}$$
$$z = z$$



Spherical to Cartesian:

$$x = \rho\sin\phi\cos\theta$$
$$y = \rho\sin\phi\sin\theta \quad \textit{(3D)}$$
$$z = \rho\cos\phi$$



PART 14: FUNCTIONS

14.1 ODD & EVEN FUNCTIONS:

Definitions:

Even: $f(-x) = f(x), \forall x$

Odd: $f(-x) = -f(x), \forall x$

Composite Functions:

Odd ± Odd = Odd
Odd ± Even = Neither
Even ± Even = Even

Odd x Odd = Even
Odd / Odd = Even
Even x Even = Even
Even / Even = Even

Even of Odd = Even
Even of Even = Even
Even of Neither = Neither
Odd of Odd = Odd
Odd of Even = Even
Odd of Neither = Neither

Basic Integration:

If f(x) is odd: $\int_{-a}^{a} f(x)dx = 0$

If f(x) is even: $\int_{-a}^{a} f(x)dx = 2\int_{0}^{a} f(x)dx$

14.2 MULTIVARIABLE FUNCTIONS:

Limit: $\lim_{(x,y)\to(0,0)}\left(f_{(x,y)}\right) = \lim_{(x,mx)\to(0,0)}\left(f_{(x,mx)}\right) = \lim_{(x)\to(0)}\left(f_{(x,mx)}\right)$

Discriminant: $D_{(x_0,y_0)} = z_{xx}z_{yy} - \left(z_{xy}\right)^2$

Critical Points: $z = f_{(x,y)}$

Solve for: $\left.\begin{matrix} z_x = 0 \\ z_y = 0 \end{matrix}\right\}$

- If the critical point (x_0,y_0) is a local maximum, then

  $D(x_0,y_0) >= 0$

  $f_{xx}(x_0,y_0) <= 0$ and $f_{yy}(x_0,y_0) <= 0$

- If $D(x_0,y_0) > 0$, and either

  $f_{xx}(x_0,y_0) < 0$ or $f_{yy}(x_0,y_0) < 0$

  then the critical point (x_0,y_0) is a local maximum.

- If the critical point (x_0,y_0) is a local minimum, then

  $D(x_0,y_0) >= 0$

  $f_{xx}(x_0,y_0) >= 0$ and $f_{yy}(x_0,y_0) >= 0$
- If $D(x_0,y_0) > 0$, and either

  $f_{xx}(x_0,y_0) > 0$ or $f_{yy}(x_0,y_0) > 0$

  then the critical point (x_0,y_0) is a local minimum.
- If the critical point (x_0,y_0) is a saddle point, then

  $D(x_0,y_0) <= 0$
- If

  $D(x_0,y_0) < 0$,

  then the critical point (x_0,y_0) is a saddle point.

14.3 FIRST ORDER, FIRST DEGREE, DIFFERENTIAL EQUATIONS:

Separable:

$$\frac{dy}{dx} = \frac{f(x)}{g(y)}$$
$$g(y)dy = f(x)dx$$
$$\int g(y)dy = \int f(x)dx$$

Linear:

$$\frac{dy}{dx} + P(x) \times y = Q(x)$$
$$I(x) = e^{\int P(x)dx}$$
$$y = \frac{1}{I(x)}\left(\int I(x) \times Q(x)dx\right)$$

Homogeneous:

$$f(\lambda x, \lambda y) = f(x, y)$$
$$\frac{dy}{dx} = f(x, y) = F\left(\frac{y}{x}\right)$$

Let $v(x) = \frac{y}{x}, \therefore \frac{dy}{dx} = v + x\frac{dv}{dx}$

$$\therefore v + x\frac{dv}{dx} = F(v)$$

$$x\frac{dv}{dx} = F(v) - v$$

$$\frac{dv}{F(v) - v} = \frac{dx}{x}$$

$$\int \frac{dv}{F(v) - v} = \int \frac{dx}{x}$$

Exact:

$$\frac{dy}{dx} = f(x, y) \to M(x, y)dx + N(x, y)dy = 0$$

If: $M_y = N_x$

When: $F_X = M \,\&\, F_Y = N$

Therefore,

$$F = \int M(x, y)dx = \Phi(x, y) + g(y)$$

$$F_Y = \frac{\partial}{\partial y}\left(\Phi + g(y)\right) = \Phi_Y + g'(y) = N$$

$$\therefore g(y) = \dots$$

So: $F(x, y) = \Phi(x, y) + g(y) = C$

Bernoulli Form:

$$\frac{dy}{dx} + P(x)y = Q(x)y^n$$

Let:

$$v = y^{1-n}$$

$$\therefore \frac{dv}{dx} = (1-n)y^{-n}\frac{dy}{dx}$$

$$\therefore \frac{dy}{dx} = \frac{y^n}{1-n}\frac{dv}{dx}$$

$$\therefore \frac{y^n}{1-n}\frac{dv}{dx} + P(x)y = Q(x)y^n$$

$$\Rightarrow \frac{1}{1-n}\frac{dv}{dx} + P(x)y^{1-n} = Q(x)$$

$$\Rightarrow \frac{dv}{dx} + (1-n)P(x)v = (1-n)Q(x)$$

This is a linear 1st ODE

$$\frac{dv}{dx} + P_2(x) \times v = Q_2(x)$$

$$I(x) = e^{\int P_2(x)dx}$$

$$v = \frac{1}{I(x)}\left(\int I(x) \times Q_2(x)dx\right)$$

$$y = \sqrt[(1-n)]{v} = \sqrt[(1-n)]{\frac{1}{I(x)}\left(\int I(x) \times Q_2(x)dx\right)} = \sqrt[(1-n)]{\frac{1}{e^{\int(1-n)P(x)dx}}\left(\int e^{\int(1-n)P(x)dx} \times (1-n)Q(x)dx\right)}$$

14.4 SECOND ORDE, FIRST DEGREE, DIFFERENTIAL EQUATIONS:

Where $a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = f(x)$

$ay'' + by' + cy = f(x)$

Greg's Lemma:

$$(D-\alpha)\left(p(x)e^{\alpha x}\right) = p'(x)e^{\alpha x}$$

$$(D-\alpha)\left(xe^{\alpha x}\right) = e^{\alpha x}$$

$$(D-\alpha)^2\left(x^2e^{\alpha x}\right) = 2e^{\alpha x}$$

$$(D-\alpha)^n\left(x^ne^{\alpha x}\right) = n!e^{\alpha x}$$

Homogeneous:

$$ay'' + by' + cy = 0$$

$$\Rightarrow am^2 + bm + c = 0$$

$$m = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

There are three possible outcomes:

1) m_1, m_2 where $m_1 \neq m_2$ $\quad \Rightarrow y_h = Ae^{m_1x} + Be^{m_2x}$ $\quad S = \{e^{m_1x}, e^{m_2x}\}$
2) m_1, m_2 where $m_1 = m_2$ $\quad \Rightarrow y_h = (A + Bx)e^{m_1x}$ $\quad S = \{e^{m_1x}, xe^{m_1x}\}$
3) $m_{1,2} = \alpha \pm \beta j$ $\quad \Rightarrow y_h = e^{\alpha x}(A\cos(\beta x) + B\sin(\beta x))$ $\quad S = \{e^{\alpha x}\cos(\beta x), e^{\alpha x}\sin(\beta x)\}$

Undetermined Coefficients

$ay'' + by' + cy = f(x)$, where $f(x)$ is in the form of

1) A polynomial
2) $\alpha\sin(kx)$
3) αe^{kt}

NB: Multiplication is OK: eg: $f(x) = 3x^3e^x$

Step 1: Solve for the homogeneous case $ay'' + by' + cy = 0$

$$\therefore y_h = c_1y_1 + c_2y_2$$

$$S = \{y_1, y_2\}$$

Step 2: Determine the spanning set

1) A polynomial $\Rightarrow y_p = A_n x^n + A_{n-1}x^{n-1} + A_1 x + A_0$ $T = \{x^n, x^{n-1}, ...\}$

2) $\alpha \sin(kx)$ $\Rightarrow y_p = A\sin(kx) + B\cos(kx)$ $T = \{\cos(kx), \sin(kx)\}$

3) αe^{kt} $\Rightarrow y_p = Ae^{kt}$ $T = \{e^{kt}\}$

NB: Multiplication is OK: eg:

$$f(x) = 3x^3 e^x$$
$$y_p = (Ae^x)(Bx^3 + Cx^2 + Dx + E)$$
$$y_p = (e^x)(Bx^3 + Cx^2 + Dx + E)$$

Step 3: Determine the form of y_p with arbitary coefficients

Case 1: When $S \cap T = \Phi$
Then y_p is a linear combination of the elements of T

Case 2: When $S \cap T \neq \Phi$ (ie: y_p is part of y_h)
Multiply each element in T by x to get a new set T' and then recheck if $S \cap T' = \Phi$. This is now a case 1, otherwise repeat the process until $S \cap T' = \Phi$

Step 4: Determine the coefficients of y_p by substituting y_p, y'_p, y''_p back into the orginal differential equation and comparing the coefficients with $f(x)$

Then,

$$y = y_h + y_{p1} + y_{p2} + y_{p3} + ...$$

If the intital values are given, go on to solve for the constants within the homogeneous part.

Variation of Parameters

Where $\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = f(x)$ and the method of undetermined coefficients is not suitable.

$$y'' + by' + cy = f(x)$$

Step 1: Solve for the homogeneous case $ay'' + by' + cy = 0$

$$\therefore y_h = c_1 y_1 + c_2 y_2$$

Step 2: Assume that there is a solution of the differential equation of the form:

$$y = v_1(x)u_1(x) + v_2(x)u_2(x) = v_1u_1 + v_2u_2$$

Such that $v_1'(x)u_1(x) + v_2'(x)u_2(x) = v_1'u_1 + v_2'u_2 = 0$

Therefore,

$$y = v_1u_1 + v_2u_2$$
$$y' = v_1'u_1 + v_1u_1' + v_2'u_2 + v_2u_2' = v_1u_1' + v_2u_2'$$
$$y'' = v_1u_1'' + v_1'u_1' + v_2'u_2' + v_2u_2''$$

Substituting into the differential equation:

$$f(x) = (v_1u_1'' + v_1'u_1' + v_2'u_2' + v_2u_2'') + b(v_1u_1' + v_2u_2') + c(v_1u_1 + v_2u_2)$$
$$f(x) = u_1''v_1 + bu_1'v_1 + cu_1v_1 + u_2''v_2 + bu_2'v_2 + cu_2v_2 + u_1'v_1' + u_2'v_2'$$
$$f(x) = (u_1'' + bu_1' + cu_1)v_1 + (u_2'' + bu_2' + cu_2)v_2 + u_1'v_1' + u_2'v_2'$$
$$f(x) = u_1'v_1' + u_2'v_2'$$

As $u_1''+bu_1'+cu_1=0$ & $u_2''+bu_2'+cu_2=0$

We know that $u_1v_1'+u_2v_2'=0$ and $u_1'v_1'+u_2'v_2'=f(x)$

Step 3:

Solving the equations above by Cramer's Rule.

$$\begin{bmatrix} u_1 & u_2 \\ u_1' & u_2' \end{bmatrix}\begin{bmatrix} v_1' \\ v_2' \end{bmatrix}=\begin{bmatrix} 0 \\ f(x) \end{bmatrix}$$

$$v_1'=\frac{\left\|\begin{bmatrix} 0 & u_2 \\ f(x) & u_2' \end{bmatrix}\right\|}{\left\|\begin{bmatrix} u_1 & u_2 \\ u_1' & u_2' \end{bmatrix}\right\|}=\frac{-f(x)u_2}{W(u_1,u_2)}$$

$$v_2'=\frac{\left\|\begin{bmatrix} u_1 & 0 \\ u_1' & f(x) \end{bmatrix}\right\|}{\left\|\begin{bmatrix} u_1 & u_2 \\ u_1' & u_2' \end{bmatrix}\right\|}=\frac{f(x)u_1}{W(u_1,u_2)}$$

Step 4:

$$v_1(x)=\int v_1'dx=\int\frac{\left\|\begin{bmatrix} 0 & u_2 \\ f(x) & u_2' \end{bmatrix}\right\|}{\left\|\begin{bmatrix} u_1 & u_2 \\ u_1' & u_2' \end{bmatrix}\right\|}dx=\int\frac{-f(x)u_2}{W(u_1,u_2)}dx$$

$$v_2(x)=\int v_2'dx=\int\frac{\left\|\begin{bmatrix} u_1 & 0 \\ u_1' & f(x) \end{bmatrix}\right\|}{\left\|\begin{bmatrix} u_1 & u_2 \\ u_1' & u_2' \end{bmatrix}\right\|}dx=\int\frac{f(x)u_1}{W(u_1,u_2)}dx$$

$$y_h=c_1u_1(x)+c_2u_2(x)$$
$$y_p=v_1(x)u_1(x)+v_2(x)u_2(x)$$
$$y=y_h+y_p$$

Euler Type

Of the form: $ax^2y''+bxy'+cy=0, x>0$

Characteristic Equation: $a(r)(r-1)+b(r)+c=0$ or $r^2+\frac{(b-1)}{a}r+\frac{c}{a}y=0$

There are three possible outcomes:

1) r_1,r_2 where $r_1\neq r_2$ $\Rightarrow y_h=C_1x^{r_1}+C_2x^{r_2}, x>0$
2) r_1,r_2 where $r_1=r_2$ $\Rightarrow y_h=C_1x^{r_1}+C_2x^{r_1}\ln(x), x>0$
3) $r_{1,2}=\alpha\pm\beta j$ $\Rightarrow y_h=x^{\alpha}(A\cos(\beta\ln(x))+B\sin(\beta\ln(x))), x>0$

Reduction of Order

Of the form: $f_1(x)y''+f_2(x)y'+f_3(x)y=0$ where a solution is known $y_1=g(x)$

Therefore, let

$$y = v(x)g(x)$$
$$\therefore y' = v'(x)g(x) + v(x)g'(x)$$
$$\therefore y'' = v''(x)g(x) + v'(x)g'(x) + v'(x)g'(x) + v(x)g''(x)$$
$$\therefore y'' = v''(x)g(x) + 2v'(x)g'(x) + v(x)g''(x)$$

Substituting Yields:

$$f_1(x)\left(v''(x)g(x) + 2v'(x)g'(x) + v(x)g''(x)\right) + f_2(x)\left(v'(x)g(x) + v(x)g'(x)\right) + f_3(x)\left(v(x)g(x)\right) = 0$$

This should cancel down to only include $v' \& v''$

$$f_1(x)\left(v''(x)g(x) + 2v'(x)g'(x)\right) + f_2(x)\left(v'(x)g(x)\right) = 0$$

Therefore, let $u = v' \& u' = v''$

$$f_1(x)\left(u'(x)g(x) + 2u(x)g'(x)\right) + f_2(x)\left(u(x)g(x)\right) = 0$$
$$f_1(x)u'(x)g(x) + 2f_1(x)u(x)g'(x) + f_2(x)u(x)g(x) = 0$$
$$u'(x)\left(f_1(x)g(x)\right) + u(x)\left(2f_1(x)g'(x) + f_2(x)g(x)\right) = 0$$
$$\frac{u'(x)}{u(x)} = \frac{-\left(2f_1(x)g'(x) + f_2(x)g(x)\right)}{\left(f_1(x)g(x)\right)}$$
$$\int \frac{u'(x)}{u(x)}du = \int \frac{-\left(2f_1(x)g'(x) + f_2(x)g(x)\right)}{\left(f_1(x)g(x)\right)}dx$$
$$\ln(u(x)) = \int \frac{-\left(2f_1(x)g'(x) + f_2(x)g(x)\right)}{\left(f_1(x)g(x)\right)}dx$$
$$u(x) = e^{\int \frac{-\left(2f_1(x)g'(x) + f_2(x)g(x)\right)}{\left(f_1(x)g(x)\right)}dx}$$
$$v'(x) = e^{\int \frac{-\left(2f_1(x)g'(x) + f_2(x)g(x)\right)}{\left(f_1(x)g(x)\right)}dx}$$
$$v(x) = \int e^{\int \frac{-\left(2f_1(x)g'(x) + f_2(x)g(x)\right)}{\left(f_1(x)g(x)\right)}dx} dx$$
$$y = \int e^{\int \frac{-\left(2f_1(x)g'(x) + f_2(x)g(x)\right)}{\left(f_1(x)g(x)\right)}dx} dx \times g(x)$$

General solution is of the form: $y = c_1 y_1 + c_2 y_2$

Note: Constants can be combined into a single constant to simplify working if required.

Power Series Solutions:

Where $P(x)\frac{d^2y}{dx^2} + Q(x)\frac{dy}{dx} + R(x)y = 0$ about a point x_0

$$P(x)y'' + Q(x)y' + R(x)y = 0$$

Case 1: About an Ordinary Point

Where $P(x_0) \neq 0$ and analytic at x_0

Assume $y = \sum_{n=0}^{\infty} a_n (x - x_0)^n,\ y' = \sum_{n=1}^{\infty} n a_n (x - x_0)^{n-1},\ y'' = \sum_{n=1}^{\infty} n(n-1) a_n (x - x_0)^{n-2}$

- Substitute into the differential equation.
- Adjust the indicies in terms of x^n
- Obtain a recurance relation of a_n in terms of a_0 & a_1

- Express a_n in terms of a_0 & a_1

$$y = \sum_{n=0}^{\infty} a_n (x - x_0)^n = a_0 y_1 + a_1 y_2$$

Case 2: About a Regular Singular Point

Where $P(x_0) = 0$, $\lim_{x \to x_0}\left((x - x_0)\frac{Q(x)}{P(x)}\right)$ exists and $\lim_{x \to x_0}\left((x - x_0)^2 \frac{R(x)}{P(x)}\right)$ exists.

For convienance, taking about $x_0 = 0$

$$x\frac{Q(x)}{P(x)} = xp(x) = \sum_{n=0}^{\infty} p_n x^n$$

$$x^2 \frac{R(x)}{P(x)} = x^2 q(x) = \sum_{n=0}^{\infty} q_n x^n$$

This is convergent on $|x| < R$

$$x^2 y'' + x(xp(x))y' + (x^2 q(x))y = 0$$

$$x^2 y'' + x\left(\sum_{n=0}^{\infty} p_n x^n\right)y' + \left(\sum_{n=0}^{\infty} q_n x^n\right)y = 0$$

$$x^2 y'' + x(p_0 + p_1 x + p_2 x^2 + \ldots)y' + (q_0 + q_1 x + q_2 x^2 + \ldots)y = 0$$

As near $x_0 = 0$, $x \to 0$, the equation behaves like

$$x^2 y'' + x p_0 y' + q_0 y = 0$$

This is an Euler form.

Indicial Equation: $r(r-1) + p_0 r + q_0 = 0$, Thus $r = r_1, r_2 \mid r_1 \geq r_2$

Therefore, $y = x^r (a_0 + a_1 x + a_2 x^2 + \ldots) = \sum_{n=0}^{\infty} a_n x^{r+n}$ for $a_0 \neq 0, x > 0$

From the indicial equation, there are three different forms of the general solution.

Form 1: $\{r_1 - r_2\} \notin \{\mathrm{Z}\}$

$$y_1 = |x - x_0|^{r_1} \sum_{n=0}^{\infty} a_n (x - x_0)^n$$

$$y_2 = |x - x_0|^{r_2} \sum_{n=0}^{\infty} b_n (x - x_0)^n$$

$$y = c_1 y_1 + c_2 y_2$$

Form 2: $r_1 = r_2$

$$y_1 = x^{r_1}\left(1 + \sum_{n=1}^{\infty} a_n r_1 x^n\right)$$

$$y_2 = y_1 \ln(x) + x^{r_1}\left(1 + \sum_{n=1}^{\infty} b_n r_1 x^n\right)$$

$$y = c_1 y_1 + c_2 y_2$$

Where the coefficients of the series are found by substituting into the differential equation.

Form 3: $\{r_1 - r_2\} \in \{\mathrm{Z}\}$

$$y_1 = x^{r_1}\left(1+\sum_{n=1}^{\infty} a_n r_1 x^n\right)$$

$$y_2 = a y_1 \ln(x) + x^{r_2}\left(1+\sum_{n=1}^{\infty} b_n r_2 x^n\right)$$

$$y = c_1 y_1 + c_2 y_2$$

Where the coefficients of the series are found by substituting into the differential equation.

Case 3: About an Irregular Singular Point

Where $P(x_0) = 0$, $\lim_{x \to x_0}\left((x - x_0)\frac{Q(x)}{P(x)}\right)$ does not exist or $\lim_{x \to x_0}\left((x - x_0)^2 \frac{R(x)}{P(x)}\right)$ does not exist.

14.5 ORDINARY DIFFERENTIAL EQUATIONS USING MATRICES:

Derivation of Methods:

$$\begin{aligned} x^{\bullet}{}_1(t) &= a_{11}x_1(t) + a_{12}x_2(t) + ... + a_{1n}x_n(t) \\ x^{\bullet}{}_2(t) &= a_{21}x_1(t) + a_{22}x_2(t) + ... + a_{2n}x_n(t) \\ &... \\ x^{\bullet}{}_n(t) &= a_{n1}x_1(t) + a_{n2}x_2(t) + ... + a_{nn}x_n(t) \end{aligned} \Rightarrow \underline{x}^{\bullet}(t) = A\underline{x}(t) \Rightarrow \underline{x}^{\bullet} = A\underline{x}$$

where $A = [a_{ij}], \underline{x} = \begin{bmatrix} x^{\bullet}{}_1(t) \\ x^{\bullet}{}_2(t) \\ ... \\ x^{\bullet}{}_n(t) \end{bmatrix} \in \Re^n$

$$\therefore \underline{x} = c_1 \underline{x}_1 + c_2 \underline{x}_2 + ... + c_n \underline{x}_n$$

$$\underline{x} = [\underline{x}_1 \mid \underline{x}_2 \mid ... \mid \underline{x}_n] \begin{bmatrix} c_1 \\ c_2 \\ ... \\ c_n \end{bmatrix}$$

$$\underline{x}(t) = \Phi(t)\underline{c}$$

Fundamental Matrix:

General Solution: $\underline{x}(t) = \Phi(t)\underline{c} = \begin{bmatrix} v_{11}e^{\lambda_1 t} & v_{12}e^{\lambda_2 t} & ... & v_{n1}e^{\lambda_n t} \\ v_{21}e^{\lambda_1 t} & v_{22}e^{\lambda_2 t} & ... & v_{n2}e^{\lambda_n t} \\ ... & ... & ... & ... \\ v_{n1}e^{\lambda_1 t} & v_{n2}e^{\lambda_2 t} & ... & v_{nn}e^{\lambda_n t} \end{bmatrix}\underline{c}$

Particular Solution: $\underline{c} = \Phi^{-1}(t_0)\underline{x}_0$

Homogeneous Solution:

General Solution:
$$\underline{x}^{\bullet} = A\underline{x}$$
$$\underline{x} = e^{At}\underline{c} = Pe^{Jt}P^{-1}\underline{c}$$

Particular Solution: $$\underline{c} = \begin{bmatrix} x_1(0) \\ x_2(0) \\ \dots \\ x_n(0) \end{bmatrix}$$

Inhomogeneous Solution:

$$\underline{x}^{\bullet}(t) = A\underline{x}(t) + \underline{u}(t)$$

General Solution: $$\underline{x}(t) = \Phi(t)\underline{c} + \Phi(t)\int \Phi^{-1}(t)\underline{u}(t)dt$$

$$\underline{x}(t) = \Phi(t)\underline{c} + \Phi(t)\int \Phi(-t)\underline{u}(t)dt$$

Particular Solution: $$\underline{c} = \begin{bmatrix} x_1(0) \\ x_2(0) \\ \dots \\ x_n(0) \end{bmatrix}$$

nth Order linear, constant coefficient ODE:

More generally, given an n-th order linear, constant coefficient ODE,

$$y^{(n)} + a_{n-1}y^{(n-1)} + a_{n-2}y^{(n-2)} + \ldots + a_2y'' + a_1y' + a_0y = 0,$$

we may put

$$x_1 = y,\ x_2 = y',\ x_3 = y'',\ \ldots,\ x_{n-1} = y^{(n-2)},\ x_n = y^{(n-1)},$$

so that

$$\begin{aligned} \dot{x}_1 &= y' = x_2 \\ \dot{x}_2 &= (y')' = y'' = x_3 \\ \dot{x}_3 &= (y'')' = y''' = x_4 \\ &\vdots \\ \dot{x}_{n-1} &= x_n \end{aligned}$$

and finally,

$$\begin{aligned} \dot{x}_n &= (y^{(n-1)})' \\ &= y^{(n)} \\ &= -a_{n-1}y^{(n-1)} - a_{n-2}y^{(n-2)} - \ldots - a_2y'' - a_1y' - a_0y \\ &= -a_0x_1 - a_1x_2 - a_2x_3 - \ldots - a_{n-2}x_{n-1} - a_{n-1}x_n \end{aligned}$$

i.e. we get a linear system $\dot{\boldsymbol{x}} = A\boldsymbol{x}$, where

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \ddots & 1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 \\ -a_0 & -a_1 & -a_2 & -a_3 & \cdots & -a_{n-2} & -a_{n-1} \end{bmatrix}.$$

Clearly, A has a special structure and it is said to be in <u>companion form</u>. Note that the characteristic equation of A is given by

$$\det(A - \lambda I) = \begin{vmatrix} -\lambda & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & -\lambda & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & -\lambda & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \ddots & 1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & -\lambda & 1 \\ -a_0 & -a_1 & -a_2 & -a_3 & \cdots & -a_{n-2} & -a_{n-1}-\lambda \end{vmatrix} = 0.$$

Taking a cofactor expansion along the first column, this becomes

$$-\lambda \underbrace{\begin{vmatrix} -\lambda & 1 & 0 & \cdots & 0 & 0 \\ 0 & -\lambda & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ddots & 1 & 0 \\ 0 & 0 & 0 & \cdots & -\lambda & 1 \\ -a_1 & -a_2 & -a_3 & \cdots & -a_{n-2} & -a_{n-1}-\lambda \end{vmatrix}}_{=(-1)^{n-1}(\lambda^{n-1}+a_{n-1}\lambda^{n-2}+\ldots+a_2\lambda+a_1)} + (-1)^n a_0 \underbrace{\begin{vmatrix} 1 & 0 & \cdots & 0 & 0 \\ -\lambda & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & -\lambda & 1 \end{vmatrix}}_{=1} = 0,$$

Using an induction argement, we can show that

$$(-1)^n \left(\lambda^n + a_{n-1}\lambda^{n-1} + a_{n-2}\lambda^{n-2} + \ldots + a_2\lambda^2 + a_1\lambda + a_0\right) = 0,$$

$$\textit{i.e.} \quad \lambda^n + a_{n-1}\lambda^{n-1} + a_{n-2}\lambda^{n-2} + \ldots + a_2\lambda^2 + a_1\lambda + a_0 = 0,$$

which is identical to the *characteristic equation* of the original ODE!

14.6 APPLICATIONS OF FUNCTIONS

Terminology:

$f : \mathbb{R} \to \mathbb{R}$ 1-D function or scalar function

$f : \mathbb{R} \to \mathbb{R}^3$ vector function

$f : \mathbb{R}^3 \to \mathbb{R}$ scalar field function

$f : \mathbb{R}^3 \to \mathbb{R}^3$ vector field function

Gradient Vector of a Scalar Field:

$$grad[f(x,y,z)] = \nabla f(x,y,z) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right) = \frac{\partial f}{\partial x}\underline{i} + \frac{\partial f}{\partial x}\underline{j} + \frac{\partial f}{\partial x}\underline{k}$$

Directional Derivatives:

For a function $f(x,y,z)$ and unit vector $\underline{u} = u_1\underline{i} + u_2\underline{j} + u_3\underline{k}, |\underline{u}| = 1$, the directional derivative of f at the point $P_0 = (x_0, y_0, z_0)$ in the domain of the function and in the direction $\underline{u}$ is:

$$D_u f(x_0,y_0,z_0) = \lim_{h \to 0} \frac{f(x_0 + hu_1, y_0 + hu_2, z_0 + hu_3) - f(x_0,y_0,z_0)}{h}$$

$$D_u f(x_0,y_0,z_0) = \nabla f(x,y,z) \bullet \underline{u}$$

Optimising the Directional Derivative:

- The function $f(x,y,z)$ increase most rapidly at any point P in its domain in the direction of the gradient vector $\nabla f(P)$. The directional derivative in this direction is $|\nabla f(P)|$. Therefore, the gradient ∇f always points in the direction of the most rapid increase of the function, which is referred to as the steepest ascent direction. Thus, if we want to maximize a function, it is important to move in the gradient direction.
- The function $f(x,y,z)$ decreases most rapidly at any point P in its domain in the negative direction of the gradient vector $\nabla f(P)$. The directional derivative in this direction is $-|\nabla f(P)|$. Therefore, the antigradient $-\nabla f$ always points in the direction of the most rapid decrease of the function, which is referred to as the steepest descent direction. Thus, if we want to minimize a function, it is important to move in the negative gradient direction.
- The rate of change is zero in the direction perpendicular to $\nabla f(P)$

14.7 ANALYTIC FUNCTIONS

If functions are analytic at a point x_0:
Analytic ± Analytic = Analytic
Analytic x Analytic = Analytic
Analytic / Analytic = Analytic

PART 15: MATRICIES

15.1 BASIC PRINICPLES:

Size = $i \times j$, i=row, j=column

$A = \lfloor a_{ij} \rfloor$

15.2 BASIC OPERTAIONS:

Addition: $A + B = \lfloor a_{ij} + b_{ij} \rfloor$

Subtraction: $A - B = \lfloor a_{ij} - b_{ij} \rfloor$

Scalar Multiple: $kA = \lfloor ka_{ij} \rfloor$

Transpose: $\lfloor A^T \rfloor_{ij} = A_{ji}$

eg: $\begin{bmatrix} 1 & 2 & 3 \\ 0 & -6 & 7 \end{bmatrix}^T = \begin{bmatrix} 1 & 0 \\ 2 & -6 \\ 3 & 7 \end{bmatrix}$

$(A + B + C + ...)^T = A^T + B^T + C^T + ...$

$(ABCD...)^T = ...D^T C^T B^T A^T$

Scalar Product: $a \bullet b = [a_1 \quad a_2 \quad a_3 \quad ...] \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ ... \end{bmatrix}$

Symmetry: $A^T = A$

Cramer's Rule:

$Ax = B$

$x_i = \dfrac{\det(A_i)}{\det(A)}$ where Ai = column i replaced by B

Least Squares Solution

In the form $A\underline{x} = \underline{b}$, $\underline{x} = (A^T A)^{-1} A^T \underline{b}$

For a linear approximation: $r_0 + r_1 x = b$

For a quadratic approximation: $r_0 + r_1 x + r_2 x^2 = b$

Etc.

When columns are not Linearly Independent: $x = A^+ b = VS^+ U^T b$

15.3 SQUARE MATRIX:

Diagonal: $\begin{bmatrix} d_{11} & 0 & 0 \\ 0 & d_{22} & 0 \\ 0 & 0 & d_{33} \end{bmatrix}$

Lower Triangle Matrix: $\begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix}$

Upper Triangle Matrix: $\begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$

15.4 DETERMINATE:

2x2: det $(A) = ad - bc$

3x3: det $(A) = aei + bfg + cdh - afh - bdi - ceg$

nxn: det $(A) = a_{11}C_{11} + a_{12}C_{12} + a_{1n}C_{1n} = \sum_{j=1}^{n} a_{1j}C_{ij} = \sum_{j=1}^{n} a_{1j}Mi_{1j} \times (-1)^{(1+j)}$

Rules:

1. If A has a row or a column of zeros, $\det(A) = 0$.

$$e.g. \begin{vmatrix} 5 & 0 & 6 & -1 \\ 0 & 0 & 8 & -2 \\ 1 & 0 & -3 & 4 \\ 3 & 0 & 0 & 1 \end{vmatrix} = 0 \text{ and } \begin{vmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 3 & -1 \end{vmatrix} = 0.$$

2. Multiply any one row of A by a scalar k to obtain A'. Then

$$\det(A') = k \det(A)$$

(makes sense when you consider taking a cofactor expansion along that row) $e.g.$ $\begin{vmatrix} 1 & 3 \\ -1 & 2 \end{vmatrix} = 5$ and $\begin{vmatrix} 3 & 9 \\ -1 & 2 \end{vmatrix} = 15$, as expected

3. Interchange any two rows in A to obtain A'. Then

$$\det(A') = -\det(A)$$

$$e.g. \begin{vmatrix} 13 & 1 \\ 2 & -1 \end{vmatrix} = -15 \text{ and } \begin{vmatrix} 2 & -1 \\ 13 & 1 \end{vmatrix} = 15, \text{ as expected.}$$

4. Add a multiple of one row in A to another to obtain A'. Then

$$\det(A') = \det(A)$$

e.g. let $A = \begin{bmatrix} 1 & 3 \\ -3 & 5 \end{bmatrix}$, then $\det(A) = 14$. Consider $\begin{bmatrix} 1 & 3 \\ -3 & 5 \end{bmatrix}$ $\mathrm{R}_2 \to \mathrm{R}_2 + 3\mathrm{R}_1$ $\sim \begin{bmatrix} 1 & 3 \\ 0 & 14 \end{bmatrix} = A'$ and $\det(A') = 14$, as expected.

Note that Rules 2, 3 and 4 deal in particular with how elementary row operation affect the determinant of a matrix. Rule 4 is particularly useful as it allows us to induce additional zeros in a matrix to simplify the determinant calculation.

<u>**Ex:**</u> Evaluate $\begin{vmatrix} 2 & -1 & 3 \\ 1 & 2 & 6 \\ -3 & 5 & 1 \end{vmatrix}$.

<u>**Soln:**</u> $\begin{vmatrix} 2 & -1 & 3 \\ 1 & 2 & 6 \\ -3 & 5 & 1 \end{vmatrix} \begin{matrix} \mathrm{R}_1 \to \mathrm{R}_1 - 2\mathrm{R}_2 \\ \\ \mathrm{R}_3 \to \mathrm{R}_3 + 3\mathrm{R}_2 \end{matrix} = \begin{vmatrix} 0^+ & -5 & -9 \\ 1^- & 2 & 6 \\ 0^+ & 11 & 19 \end{vmatrix} = -\begin{vmatrix} -5 & -9 \\ 11 & 19 \end{vmatrix} = -4.$

5. If one row in A is a scalar multiple of another, then $\det(A) = 0$. This makes sense, since we can then apply an e.r.o. to get a zero row before applying Rule 1. *e.g.* $\begin{vmatrix} 3 & -1 & 4 & 7 \\ 2 & 2 & 3 & -1 \\ -3 & 1 & -4 & -7 \\ 1 & 6 & 2 & 1 \end{vmatrix} = 0$, since $\mathrm{R}_3 = -\mathrm{R}_1$.

6. $\det(A^\top) = \det(A)$. This rule basically allows us to apply Rules 1-5 to columns as well as rows.

<u>**Ex:**</u> $\begin{vmatrix} 2 & 1 & 0 & -1 \\ -5 & 0 & 4 & 2 \\ 1 & -3 & 0 & 4 \\ 0 & 0 & -1 & -2 \end{vmatrix} \mathrm{R}_3 \to \mathrm{R}_3 + 3\mathrm{R}_1 = \begin{vmatrix} 2^+ & 1^- & 0 & -1 \\ -5 & 0^+ & 4 & 2 \\ 7 & 0^- & 0 & 1 \\ 0 & 0^+ & 1 & -2 \end{vmatrix} = -\begin{vmatrix} -5 & 4 & 2 \\ 7 & 0 & 1 \\ 0 & -1 & -2 \end{vmatrix} \mathrm{C}_3 \to \mathrm{C}_3 - 2\mathrm{C}_2$

$$= -\begin{vmatrix} -5^+ & 4 & -6 \\ 7^- & 0 & 1 \\ 0^+ & -1^- & 0^+ \end{vmatrix} = -(-(-1))\begin{vmatrix} -5 & -6 \\ 7 & 1 \end{vmatrix} = -(-5+42) = -37.$$

7. $\det(kA) = k^n \det(A)$ (This is simply an expanded version of Rule 2.)

8. $\det(AB) = \det(A)\det(B)$ This is not an obvious rule, but a very important one which is used frequently in practice. *e.g.* let $A = \begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 2 \\ 3 & 1 \end{bmatrix}$. Then $\det(A) = 2 - 12 = -10$, $\det(B) = -1 - 6 = -7$, $AB = \begin{bmatrix} 7 & 7 \\ -1 & 9 \end{bmatrix}$ and $\det(AB) = 63 - (-7) = 70 = \det(A)\det(B)$, as expected.

9. An upper triangular matrix is square with all entries below the main diagonal equal to zero. A lower triangular matrix is square with all entries above the main diagonal equal to zero. Finally, the determinant of a lower triangular or upper triangular matrix is the product of all the diagonal elements. For example,

$$\text{(i)} \begin{vmatrix} 1 & 2 & 3 \\ 0 & 4 & 2 \\ 0 & 0 & 2 \end{vmatrix} = (1)\begin{vmatrix} 4 & 2 \\ 0 & 2 \end{vmatrix} = (1)((4)(2)-(0)(2)) = (1)(4)(2) = 8.$$

$$\text{(ii)} \begin{vmatrix} 2 & 0 & 0 & 0 \\ 4 & 1 & 0 & 0 \\ 3 & 0 & 5 & 0 \\ -1 & 1 & 2 & 4 \end{vmatrix} = (2)\begin{vmatrix} 1 & 0 & 0 \\ 0 & 5 & 0 \\ 1 & 2 & 4 \end{vmatrix} = (2)(1)\begin{vmatrix} 5 & 0 \\ 2 & 4 \end{vmatrix} = (2)(1)(5)(4) = 40.$$

In particular, note that the identity matrix is both upper and lower triangular, so $\det(I) = 1^n = 1$ for any order n.

10. A square matrix A is invertible if and only if $\det(A) \neq 0$. This is the most important rule of determinants.

From Rule 10, it follows that:

(i) $\det(A) = 0$ shows that A is singular, *i.e.* A^{-1} does not exist. Compare this with the case of a scalar a. If $|a| = 0$ (*i.e.* if $a = 0$), then $a^{-1} = \frac{1}{a}$ also does not exist.

$\det(A) \neq 0$ shows that A is non-singular, *i.e.* A^{-1} does exist.

(ii) Note that if A is non-singular, then $AA^{-1} = I$. Hence,

$$\begin{aligned} &i.e. & \det(AA^{-1}) &= \det(I) \\ &i.e. & \det(A)\det(A^{-1}) &= 1 \\ &i.e. & \det(A^{-1}) &= \frac{1}{\det(A)} \end{aligned}$$

15.5 INVERSE

2x2:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

3x3:

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}^{-1} = \frac{1}{aei-afh-bdi+bfg+cdh-ceg}\begin{bmatrix} ei-fh & ch-bi & bf-ce \\ fg-di & ai-cg & cd-af \\ dh-eg & bg-ah & ae-bd \end{bmatrix}$$

Minor: Mij = Determinate of Sub matrix which has deleted row i and column j

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

$$M21 = \left|\begin{bmatrix} b & c \\ h & f \end{bmatrix}\right|$$

Cofactor: $Cij = Mij \times (-1)^{(i+j)}$

Adjoint Method for Inverse:

$$adj(A) = C^T$$

$$A^{-1} = \frac{1}{\det(A)} adj(A)$$

<u>Ex:</u> Find the adjoint of $A = \begin{bmatrix} 1^{+} & 2^{-} & 1^{+} \\ -1^{-} & 0^{+} & 2^{-} \\ 1^{+} & 1^{-} & 0^{+} \end{bmatrix}$.

<u>Soln:</u> $C_{11} = +\begin{vmatrix} 0 & 2 \\ 1 & 0 \end{vmatrix} = -2$, $C_{12} = -\begin{vmatrix} -1 & 2 \\ 1 & 0 \end{vmatrix} = 2$, $C_{13} = +\begin{vmatrix} -1 & 0 \\ 1 & 1 \end{vmatrix} = -1$, $C_{21} = -\begin{vmatrix} 2 & 1 \\ 1 & 0 \end{vmatrix} = 1$, $C_{22} = +\begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} = -1$, $C_{23} = -\begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix} = 1$, $C_{31} = +\begin{vmatrix} 2 & 1 \\ 0 & 2 \end{vmatrix} = 4$, $C_{32} = -\begin{vmatrix} 1 & 1 \\ -1 & 2 \end{vmatrix} = -3$, $C_{33} = +\begin{vmatrix} 1 & 2 \\ -1 & 0 \end{vmatrix} = 2$.

Hence, $C = \begin{bmatrix} -2 & 2 & -1 \\ 1 & -1 & 1 \\ 4 & -3 & 2 \end{bmatrix}$ and $\text{adj}\,(A) = C^T = \begin{bmatrix} -2 & 1 & 4 \\ 2 & -1 & -3 \\ -1 & 1 & 2 \end{bmatrix}$.

Left Inverse:

$$CA = I$$

$$C = \left(A^T A\right)^{-1} A^T$$

(when rows(A)>columns(A))

Right Inverse:

$$AC = I$$

$$C = A^T \left(AA^T\right)^{-1}$$

(when rows(A)<columns(A))

Pseudo inverse:

For any matrix A, dim(A) = n x m.
A^+ is the pseudo inverse. dim(A^+) = m x n

$$A = USV^T, A^+ = VS^+U^T$$

$$\dim(A) = m \times n; \dim(A^+) = n \times m$$

$$A \times A^+ \times A = A$$

$$A^+ \times A \times A^+ = A^+$$

$$\left(A \times A^+\right)^T = A \times A^+$$

$$\left(A^+ \times A\right)^T = A^+ \times A$$

15.6 LINEAR TRANSFORMATION

Axioms for a linear transformation:

If $F(\underline{u}+\underline{v})=F(\underline{u})+F(\underline{v})$ [Preserves Addition]

And $F(\lambda\underline{u})=\lambda F(\underline{u})$ [Preserves Scalar Multiplication]

Transition Matrix:

The matrix that represents the linear transformation

$T(\underline{v})=c_1T(v_1)+c_2T(v_2)+...+c_nT(v_n)$

$T(x)=Ax$

$A=[T(\underline{e}_1)\,|\,T(\underline{e}_2)\,|\,...\,|\,T(\underline{e}_3)]$ (With m columns and n rows)

$(T:V\rightarrow W, \dim(V)=m, \dim(W)=n)$

Zero Transformation:

$T(\underline{v})=0, \forall \underline{v}\varepsilon V$

Identity Transformation:

$T(\underline{v})=\underline{v}, \forall \underline{v}\varepsilon V$

15.7 COMMON TRANSITION MATRICIES

Rotation (Clockwise): $$\begin{bmatrix}x'\\y'\end{bmatrix}=\begin{bmatrix}\cos\theta & \sin\theta\\-\sin\theta & \cos\theta\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}$$

Rotation (Anticlockwise): $$\begin{bmatrix}x'\\y'\end{bmatrix}=\begin{bmatrix}\cos\theta & -\sin\theta\\\sin\theta & \cos\theta\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}$$

Scaling: $$\begin{bmatrix}x'\\y'\end{bmatrix}=\begin{bmatrix}s_x & 0\\0 & s_y\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}$$

Shearing (parallel to x-axis): $$\begin{bmatrix}x'\\y'\end{bmatrix}=\begin{bmatrix}1 & k\\0 & 1\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}$$

Shearing (parallel to y-axis): $$\begin{bmatrix}x'\\y'\end{bmatrix}=\begin{bmatrix}1 & 0\\k & 1\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}$$

15.8 EIGENVALUES AND EIGENVECTORS

Definitions: All solutions of $Ax=\lambda x$

Eigenvalues: All solutions of λ of det(A-λI)=0

Eigenvectors: General solution of [A-λI][X]=0 (ie: the nullspace)

Characteristic Polynomial: The function $p(\lambda)=\det(A-\lambda I)$

<u>Result 1:</u> *Let A be an $n\times n$ matrix and let $p(\lambda)$ be its characteristic polynomial. Suppose that λ is an eigenvalue of A with corresponding eigenvector $\boldsymbol{x}$. Then:*

(i) *The leading coefficient of $p(\lambda)$ is $(-1)^n$.*

(ii) *The constant term of $p(\lambda)$ is $\det(A)$.*

(iii) *λ^k is an eigenvalue of A^k with corresponding eigenvector $\boldsymbol{x}$.*

(iv) *If A is invertible, then $\lambda\neq 0$ and $\frac{1}{\lambda}$ is an eigenvalue of A^{-1} with corresponding eigenvector $\boldsymbol{x}$.*

Algebraic Multiplicity: The number of times a root is repeated for a given eigenvalue.

$\sum$ of all algebraic multiplicity = degree of the characteristic polynomial.

Geometric Multiplicity: The number of linearly independent eigenvectors you get from a given eigenvalue.

Transformation:

$T:V \to V$

$T(x) = \lambda x$

The same process for an ordinary matrix is used.

Linearly Independence: The set of eigenvectors for distinct eigenvalues is linearly independent.

Digitalization: For a nxn matrix with n distinct eigenvalues; if and only if there are n Linearly Independent Eigenvectors:

$D = P^{-1}AP$

Where $P = [P_1 | P_2 | ... | P_n]$, P_n is an eigenvector.

$$D = \begin{bmatrix} \lambda_1 & 0 & 0 & ... & 0 \\ 0 & \lambda_2 & 0 & ... & 0 \\ 0 & 0 & \lambda_3 & ... & 0 \\ ... & ... & ... & ... & 0 \\ 0 & 0 & 0 & 0 & \lambda_n \end{bmatrix}$$

<u>**Result 7:**</u> *If $B = C^{-1}AC$, then*

(i) *the eigenvalues of B are the same as those of A.*

(ii) *each eigenvalue of A or B has the same algebraic and geometric multiplicity, regardless of which matrix they correspond to.*

(iii) *if x is an eigenvector of A corresponding to an eigenvalue λ, then $C^{-1}x$ is an eigenvector of B corresponding to λ.*

Cayley-Hamilton Theorem: Every matrix satisfies its own polynomial:

$P(\lambda) = a_n\lambda^n + a_{n-1}\lambda^{n-1} ... + a_1\lambda + a_0 = 0$

$P(\lambda) = a_nA^n + a_{n-1}A^{n-1} ... + a_1A + a_0 = 0$

Orthonormal Set: The orthonomal basis of a matrix A can be found with $P = [P_1 | P_2 | ... | P_n]$, the orthonormal set will be

$$B = \left\{ \frac{P_1}{\|P_1\|}, \frac{P_2}{\|P_2\|}, ... \frac{P_n}{\|P_n\|} \right\}$$

QR Factorisation:

$A = [u_1 | u_2 | ... | u_n] = QR$

$\dim(A) = n \times k, k \le n$

All columns are Linearly Independent

$Q = [v_1 | v_2 | ... | v_n]$ by the Gram-Schmidt Process

$$R = \begin{bmatrix} \|q_1\| & u_2^T v_1 & u_3^T v_1 & u_4^T v_1 & \dots & u_k^T v_1 \\ 0 & \|q_2\| & u_3^T v_2 & u_4^T v_2 & \dots & u_k^T v_2 \\ 0 & 0 & \|q_3\| & u_4^T v_3 & \dots & u_k^T v_3 \\ 0 & 0 & 0 & \|q_4\| & \dots & u_k^T v_4 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & \|q_k\| \end{bmatrix}$$

$$R = \begin{bmatrix} \|q_1\| & u_2 \bullet v_1 & u_3 \bullet v_1 & u_4 \bullet v_1 & \dots & u_k \bullet v_1 \\ 0 & \|q_2\| & u_3 \bullet v_2 & u_4 \bullet v_2 & \dots & u_k \bullet v_2 \\ 0 & 0 & \|q_3\| & u_4 \bullet v_3 & \dots & u_k \bullet v_3 \\ 0 & 0 & 0 & \|q_4\| & \dots & u_k \bullet v_4 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & \|q_k\| \end{bmatrix}$$

15.9 JORDAN FORMS

Generalised Diagonlisation:

$$P^{-1}AP = J$$
$$A = PJP^{-1}$$

Jordan Block:

$$J_B = \begin{bmatrix} \lambda & 1 & 0 & \dots & 0 & 0 \\ 0 & \lambda & 1 & \dots & 0 & 0 \\ 0 & 0 & \lambda & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & \lambda & 1 \\ 0 & 0 & 0 & \dots & 0 & \lambda \end{bmatrix}$$

Jordan Form:

$$J = \begin{bmatrix} J_1 & 0 & \dots & 0 \\ 0 & J_2 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & J_n \end{bmatrix}$$

Algebraic Multiplicity: The number of times λ appears on main diagonal

Geometric Multiplicity: The number of times λ appears on main diagonal without a 1 directly above it

Generalised Chain: $= \{u_m, u_{m-1}, \dots, u_2, u_1\}$, where u_1 is an eigenvector

$$u_k = (A - \lambda I)u_{k+1}$$
$$u_{k+1} = [A - \lambda I \mid u_k]$$

$P = [P_1 \mid P_2 \mid \dots \mid P_m \mid \dots]$, for every eigenvector of A

Powers:

$$A^k = PJ^kP^{-1}$$

$$J^k = \begin{bmatrix} J_1 & 0 & ... & 0 \\ 0 & J_2 & ... & 0 \\ ... & ... & ... & ... \\ 0 & 0 & ... & J_n \end{bmatrix}^k = \begin{bmatrix} J_1{}^k & 0 & ... & 0 \\ 0 & J_2{}^k & ... & 0 \\ ... & ... & ... & ... \\ 0 & 0 & ... & J_n{}^k \end{bmatrix}$$

$$J_B{}^k = \begin{bmatrix} \lambda^k & \binom{k}{1}\lambda^{k-1} & \binom{k}{2}\lambda^{k-2} & ... \\ 0 & \lambda^k & \binom{k}{1}\lambda^{k-1} & ... \\ 0 & 0 & \lambda^k & ... \\ ... & ... & ... & ... \end{bmatrix}$$

15.12 SINGULAR VALUE DECOMPOSITION

Fundamentally: $A = USV^T$, U and V are orthogonal

Size:

$$\dim(A) = m \times n$$
$$\dim(U) = m \times m$$
$$\dim(S) = m \times n$$
$$\dim(V) = n \times n$$

Pseudo inverse: $A^+ = VS^+U^T$

Procedure:

STEP 1: Find the symmetric matrix A^TA. This has orthogonal diagonalisation.

STEP 2: Find a set of eigenvalues of A^TA. These are all non-negative.
$\lambda = a, b, c, d...$

STEP 3: Arrange eigenvalues in decreasing order
$\lambda_1 > \lambda_2 > \lambda_3 > ...$

STEP 4: Find σ_i
$\sigma_i = \sqrt{\lambda_i}$
$\therefore \sigma_1 > \sigma_2 > \sigma_3 > ...$

STEP 5: Find S
dim(S) = m x n
$S = \begin{bmatrix} S_1 & 0 \\ 0 & 0 \end{bmatrix}$ where S_1 is a matrix with the diagonal equal to $\sigma_1, \sigma_2, \sigma_3, ...$ and all other elements 0.

STEP 6: Find a set of eigenvectors of A^TA

$$= p_1, p_2, p_3, \ldots$$

STEP 7: Normalise each eigenvector

$$v_i = \hat{p}_i = \frac{p_i}{\|p_i\|}$$

STEP 8: Form V

$$V = [v_1 \,|\, v_2 \,|\, v_3 \,|\, \ldots]$$

STEP 9: Generate first columns of u corresponding to non-zero eigenvalues

$$x_i = \frac{1}{\sigma_i} A v_i$$

STEP 10: Generate remaining columns of U such that it is an orthogonal square matrix.

$$\begin{bmatrix} x_1 \\ x_2 \\ \ldots \\ x_a \end{bmatrix} x_{a+1} = 0$$

(ie: x_{a+1} belongs to the nullspace of the matrix that is made from the found vectors of x arranged in rows.)

STEP 11: Normalise each vector of U

$u_i = \frac{x_i}{\|x_i\|}$, this only needs to be done for x_{a+1} onwards

STEP 12: Form U

$$U = [u_1 \,|\, u_2 \,|\, u_3 \,|\, \ldots]$$

15.11 COMPLEX MATRICIS:

Conjugate Transpose:

$$A^* = \overline{A^T}$$

$$A^{**} = A$$

$$(A+B)^* = A^* + B^*$$

$$(zA)^* = \bar{z}A^*$$

$$(AB)^* = B^* A^*$$

Hermitian Matrix: (Similar to Symmetric Matricis in the real case)
A square matrix such that $A^*=A$
Eigenvalues of A are purely real
Eigenvectors from distinct eigenvalues are orthogonal. This leads to a unitary digitalisation of the Hermitian matrix.
These are normal

Skew-Hermitian:

A square matrix such that $A^*=-A$
Eigenvalues of A are purely imaginary
Eigenvectors from distinct eigenvalues are orthogonal.

If A is Skew-Hermitian, iA is normal as: $(iA)^* = \bar{i}A^* = (-i)(-A) = iA$
These are normal

Unitary Matrix: (Similar to Orthogonal Matricis in the real case)
A square matrix such that $A^*A=I$
Columns of A form an orthonormal set of vectors
Rows of A from an orthonormal set of vectors

Normal Matrix:

Where $AA^* = A^*A$
These will have unitary diagonalisation
All Hermition and Skew-Hermitian matricis are normal ($A^*A = AA = AA^*$)

Diagonalisation:

For a nxn matrix with n distinct eigenvalues; if and only if there are n Linearly Independent Eigenvectors:

$D = P^{-1}AP$

Where $P = [P_1 \mid P_2 \mid \ldots \mid P_n]$, P_n is an eigenvector.

$$D = \begin{bmatrix} \lambda_1 & 0 & 0 & \ldots & 0 \\ 0 & \lambda_2 & 0 & \ldots & 0 \\ 0 & 0 & \lambda_3 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & 0 \\ 0 & 0 & 0 & 0 & \lambda_n \end{bmatrix}$$

If A is Hermitian, $D = P^{-1}AP = P^*AP$ as P are an orthonormal set of vectors.

Spectral Theorem:

For a nxn Normal matrix and eigenvectors form an orthonormal set

$P = [P_1 \mid P_2 \mid \ldots \mid P_n]$

$A = \lambda_1 P_1 P_1^* + \lambda_2 P_2 P_2^* + \ldots + \lambda_n P_n P_n^*$

Therefore, A can be represented as a sum of n matricis, all of rank 1.
Therefore, A can be approximated as a sum of the dominant eigenvalues

15.12 NUMERICAL COMPUTATIONS:

Rayleigh Quotient:

$$R(\boldsymbol{x}) = \frac{\boldsymbol{x}^\top A \boldsymbol{x}}{\boldsymbol{x}^\top \boldsymbol{x}}$$

if (λ;*v*) is an eigenvalue/eigenvector pair of *A*, then

$$R(\boldsymbol{v}) = \frac{\boldsymbol{v}^\top A \boldsymbol{v}}{\boldsymbol{v}^\top \boldsymbol{v}} = \frac{\boldsymbol{v}^\top (\lambda \boldsymbol{v})}{\boldsymbol{v}^\top \boldsymbol{v}} = \lambda \frac{\boldsymbol{v}^\top \boldsymbol{v}}{\boldsymbol{v}^\top \boldsymbol{v}} = \lambda,$$

In particular, suppose that A is a symmetric matrix. Then we know that A can be orthogonally diagonalized, *i.e.* there exists an orthogonal matrix Q such that

$$Q^\top AQ = D = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n),$$

where $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the (real) eigenvalues of A. Now, without loss of generality, suppose that λ_1 is the largest of all the eigenvalues, *i.e.* $\lambda_1 \geq \lambda_i$, for all $i = 2, 3, \ldots, n$.

Let $\boldsymbol{x}$ be any vector in $I\!R^n$ and let $\boldsymbol{s} = Q^\top \boldsymbol{x} = \begin{bmatrix} s_1 \\ s_2 \\ \vdots \\ s_n \end{bmatrix} \in I\!R^n$, so that $\boldsymbol{x} = Q\boldsymbol{s}$. Then

$$R(\boldsymbol{x}) = \frac{\boldsymbol{x}^\top A\boldsymbol{x}}{\boldsymbol{x}^\top \boldsymbol{x}} = \frac{\boldsymbol{s}^\top Q^\top AQ\boldsymbol{s}}{\boldsymbol{s}^\top Q^\top Q\boldsymbol{s}} = \frac{\boldsymbol{s}^\top D\boldsymbol{s}}{\boldsymbol{s}^\top \boldsymbol{s}} = \frac{\lambda_1 s_1^2 + \lambda_2 s_2^2 + \ldots + \lambda_n s_n^2}{s_1^2 + s_2^2 + \ldots + s_n^2}.$$

Since λ_1 is the largest eigenvalue,

$$R(\boldsymbol{x}) \leq \frac{\lambda_1 s_1^2 + \lambda_1 s_2^2 + \ldots + \lambda_1 s_n^2}{s_1^2 + s_2^2 + \ldots + s_n^2}, \quad i.e.\ R(\boldsymbol{x}) \leq \lambda_1, \text{ for all } \boldsymbol{x} \in I\!R^n.$$

i.e. For a real symmetric matrix, the maximum value of Rayleigh's quotient is equal to the largest eigenvalue of A. This fact can sometimes be used to find the largest eigenvalue of a real symmetric matrix. (Note that the above argument is not valid if A is not symmetric.)

Ex 1: Find the largest eigenvalue of $A = \begin{bmatrix} 1 & -2 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ and the corresponding eigenvector.

Soln: Let $\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, then

$$\boldsymbol{x}^\top A\boldsymbol{x} = [x_1\ x_2\ x_3] \begin{bmatrix} 1 & -2 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = [x_1\ x_2\ x_3] \begin{bmatrix} x_1 - 2x_2 \\ -2x_1 + x_2 \\ 2x_3 \end{bmatrix} = x_1^2 - 4x_1x_2 + x_2^2 + 2x_3^2.$$

We have

$$\begin{aligned} R(\boldsymbol{x}) &= \frac{\boldsymbol{x}^\top A\boldsymbol{x}}{\boldsymbol{x}^\top \boldsymbol{x}} \\ &= \frac{x_1^2 - 4x_1x_2 + x_2^2 + 2x_3^2}{x_1^2 + x_2^2 + x_3^2} \\ &= \frac{3x_1^2 + 3x_2^2 - (2x_1^2 + 4x_1x_2 + 2x_2^2) + 2x_3^2}{x_1^2 + x_2^2 + x_3^2} \\ &= \frac{3x_1^2 + 3x_2^2 + 3x_3^2 - 2(x_1 + x_2)^2 - x_3^2}{x_1^2 + x_2^2 + x_3^2} \\ &= 3 - \frac{(2(x_1 + x_2)^2 + x_3^2)}{x_1^2 + x_2^2 + x_3^2} \\ &\leq 3, \end{aligned}$$

for all x_1, x_2 and x_3. In other words, the maximum eigenvalue is $\lambda = 3$. Note that $R(\boldsymbol{x}) = 3$ only if $x_3 = 0$ and $x_1 + x_2 = 0$, *i.e.* a corresponding eigenvector is $\boldsymbol{x} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$

Power method:

If A is a nxn matrix with Linearly Independent Eigenvectors, and distinct eigenvectors arranged such that: $|\lambda_1| \ge |\lambda_2| \ge \ldots \ge |\lambda_n|$ and the set of eigenvectors are: $\{v_1, v_2, \ldots, v_n\}$

Any vector "w" can be written as:

$$w_0 = c_1 v_1 + c_2 v_2 + \ldots + c_n v_n$$

$$w_1 = Aw_0 = c_1 Av_1 + c_2 Av_2 + \ldots + c_n Av_n = c_1 \lambda_1 v_1 + c_2 \lambda_2 v_2 + \ldots + c_n \lambda_n v_n$$

$$w_s = Aw_{s-1} = c_1 \lambda_1^s v_1 + c_2 \lambda_2^s v_2 + \ldots + c_n \lambda_n^s v_n = \lambda_1 \left(c_1 v_1 + c_2 \left(\frac{\lambda_2}{\lambda_1} \right)^s v_2 + \ldots + c_n \left(\frac{\lambda_n}{\lambda_1} \right)^s v_n \right)$$

As $\left| \frac{\lambda_i}{\lambda_1} \right| < 1, \ \lim_{s \to \infty} \left(\left(\frac{\lambda_i}{\lambda_1} \right)^s \right) = 0$

$$\therefore w_s \to c_1 \lambda_1^s v_1$$

Appling this with the Rayleigh Quotient:

$$w_s = A\left(\frac{w_{s-1}}{|w_{s-1}|} \right), \lambda = R(w_s), w_0 \text{ can be any vector usually } \begin{bmatrix} 1 \\ 0 \\ \ldots \end{bmatrix}$$

15.13 POWER SERIES:

$$e^{At} = \left(I + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \ldots + \frac{A^k}{k!} + \ldots \right)^t = Pe^{Jt}P^{-1}$$

$$e^{Dt} = \left(I + D + \frac{D^2}{2!} + \frac{D^3}{3!} + \ldots + \frac{D^k}{k!} + \ldots \right)^t = \begin{bmatrix} e^{\lambda_1 t} & 0 & \ldots & 0 \\ 0 & e^{\lambda_2 t} & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & e^{\lambda_n t} \end{bmatrix}$$

$$e^{Jt} = \left(I + J + \frac{J^2}{2!} + \frac{J^3}{3!} + \ldots + \frac{J^k}{k!} + \ldots \right)^t = \begin{bmatrix} e^{J_1 t} & 0 & \ldots & 0 \\ 0 & e^{J_2 t} & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & e^{J_n t} \end{bmatrix}$$

$$e^{J_n t} = \left(I + J_n + \frac{{J_n}^2}{2!} + \frac{{J_n}^3}{3!} + \ldots + \frac{{J_n}^k}{k!} + \ldots \right)^t = \begin{bmatrix} e^{\lambda t} & te^{\lambda t} & \frac{t^2}{2!} e^{\lambda t} & \ldots \\ 0 & e^{\lambda t} & te^{\lambda t} & \ldots \\ 0 & 0 & e^{\lambda t} & \ldots \\ \ldots & \ldots & \ldots & \ldots \end{bmatrix}$$

PART 16: VECTORS

16.1 BASIC OPERATIONS:

Addition: $\underline{a}+\underline{b}=\begin{bmatrix} a_1+b_1 \\ a_2+b_2 \\ a_3+b_3 \end{bmatrix}$

Subtraction: $\underline{a}-\underline{b}=\begin{bmatrix} a_1-b_1 \\ a_2-b_2 \\ a_3-b_3 \end{bmatrix}$

Equality: $\underline{a}=\underline{b} \Leftrightarrow a_1=b_1, a_2=b_2, a_3=b_3$

$k\underline{a}+l\underline{b}=\lambda\underline{a}+\mu\underline{b} \Rightarrow k=\lambda, l=\mu$

Scalar Multiplication: $k\underline{a}=\begin{bmatrix} ka_1 \\ ka_2 \\ ka_3 \end{bmatrix}$

Parallel: $\underline{a}=k\underline{b} \Leftrightarrow \underline{a}\|\underline{b}$

Magnitude: $|\underline{a}|=\sqrt{(\underline{a}_1)^2+(\underline{a}_2)^2+(\underline{a}_3)^2}$

Unit Vector: $\hat{\underline{a}}=\frac{\underline{a}}{|\underline{a}|}$

Zero Vector: A vector with no magnitude and no specific direction

Dot Product: $\underline{a}\bullet\underline{b}=|\underline{a}|\cdot|\underline{b}|\cdot\cos\theta$

$\underline{a}\bullet\underline{b}=\underline{a}_1\underline{b}_1+\underline{a}_2\underline{b}_2+\underline{a}_3\underline{b}_3$

Angle Between two Vectors: $\cos\theta=\frac{\underline{a}\bullet\underline{b}}{|\underline{a}|\cdot|\underline{b}|}$

$$\cos\theta=\frac{\underline{a}_1\underline{b}_1+\underline{a}_2\underline{b}_2+\underline{a}_3\underline{b}_3}{\left(\sqrt{\underline{a}_1{}^2+\underline{a}_2{}^2+\underline{a}_3{}^2}\right)\cdot\left(\sqrt{\underline{b}_1{}^2+\underline{b}_2{}^2+\underline{b}_3{}^2}\right)}$$

Angle of a vector in 3D: $\hat{\underline{a}}=\begin{bmatrix} \frac{a_1}{|\underline{a}|} \\ \frac{a_2}{|\underline{a}|} \\ \frac{a_3}{|\underline{a}|} \end{bmatrix}=\begin{bmatrix} \cos(\alpha) \\ \cos(\beta) \\ \cos(\gamma) \end{bmatrix}$

Perpendicular Test: $\underline{a}\bullet\underline{b}=0$

Scalar Projection: a onto b: $P = \underline{a} \bullet \hat{\underline{b}}$

Vector Projection: a onto b: $\underline{P} = \left(\underline{a} \bullet \hat{\underline{b}} \right) \hat{\underline{b}} = \frac{1}{|\underline{b}|^2} (\underline{a} \bullet \underline{b}) \underline{b}$

Cross Product:

$$\underline{a} \times \underline{b} = \langle \underline{a}_2 \underline{b}_3 - \underline{a}_3 \underline{b}_2, \underline{a}_3 \underline{b}_1 - \underline{a}_1 \underline{b}_3, \underline{a}_1 \underline{b}_2 - \underline{a}_2 \underline{b}_1 \rangle$$

$$\underline{a} \times \underline{b} = |\underline{a}| \cdot |\underline{b}| \cdot \sin\theta \cdot \underline{n}$$

$$|\underline{a} \times \underline{b}| = |\underline{a}| \cdot |\underline{b}| \cdot \sin\theta$$

$$\underline{a} \times \underline{b} = -\underline{b} \times \underline{a}$$

$$\underline{a} \cdot (\underline{b} \times \underline{c}) = \underline{b} \cdot (\underline{c} \times \underline{a}) = \underline{c} \cdot (\underline{a} \times \underline{b})$$

$$\underline{a} \times \underline{b} = \det \begin{bmatrix} \underline{i} & \underline{j} & \underline{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{bmatrix} = \underline{i} \left(\det \begin{bmatrix} a_2 & a_3 \\ b_2 & b_3 \end{bmatrix} \right) - \underline{j} \left(\det \begin{bmatrix} a_1 & a_3 \\ b_1 & b_3 \end{bmatrix} \right) + k \left(\det \begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix} \right)$$

16.2 Lines

$\underline{r} = \underline{a} + \lambda \underline{b}$, where a is a point on the line, and b is a vector parallel to the line

$$x = a_1 + \lambda b_1$$

$$y = a_2 + \lambda b_2$$

$$z = a_3 + \lambda b_3$$

$$\lambda = \frac{x - a_1}{b_1} = \frac{y - a_2}{b_2} = \frac{z - a_3}{b_3}$$

16.3 Planes

Generally:

$$\underline{n} \bullet \overrightarrow{AR} = 0$$

$$\underline{n} \bullet \underline{r} = \underline{n} \bullet \underline{a}$$

$$\underline{n} \bullet \underline{r} = k$$

Where: $\underline{n} = \langle a, b, c \rangle$ & $\underline{r} = \langle x, y, z \rangle$: $ax + by + cz = k$

Tangent Plane:

We define the tangent plane at a point $P(x_0, y_0, z_0)$ on a surface S given by the equation $f(x, y, z) = c$ as a plane which contains all tangent lines at P to curves in S through P.
Therefore,

$$0 = \nabla f(x_0, y_0, z_0) \bullet \overrightarrow{PQ}$$

$$= \langle f_x(x_0, y_0, z_0), f_y(x_0, y_0, z_0), f_z(x_0, y_0, z_0) \rangle \bullet \langle x - x_0, y - y_0, z - z_0 \rangle$$

Normal Line:

The normal line to S at P is the line passing through P and perpendicular to the tangent plan. The equation of the normal line is:

$$\frac{x-x_0}{f_x(x_0,y_0,z_0)}=\frac{y-y_0}{f_y(x_0,y_0,z_0)}=\frac{z-z_0}{f_z(x_0,y_0,z_0)}.$$

Note that

$$f_x(x_0,y_0,z_0)=\frac{\partial f}{\partial x}(x_0,y_0,z_0)$$
$$f_y(x_0,y_0,z_0)=\frac{\partial f}{\partial y}(x_0,y_0,z_0)$$
$$f_z(x_0,y_0,z_0)=\frac{\partial f}{\partial z}(x_0,y_0,z_0).$$

16.4 Closest Approach

Two Points: $d=\left|\overline{PQ}\right|$

Point and Line: $d=\left|\overline{PQ}\times\hat{\underline{a}}\right|$

Point and Plane: $d=\left|\overline{PQ}\bullet\hat{\underline{n}}\right|$

Two Skew Lines: $d=\left|\overline{PQ}\bullet\hat{\underline{n}}\right|=\left|\overline{PQ}\bullet(\underline{a}\times\underline{b})\right|$

Solving for t: $[\underline{r}_b(t)-\underline{r}_a(t)]\bullet[\underline{v}_b-\underline{v}_a]=0$
$[{}_a\underline{r}_b(t)]\bullet[{}_a\underline{v}_b]=0$

16.5 Geometry

Area of a Triangle: $A=\frac{|AB\times AC|}{2}$

Area of a Parallelogram: $A=|AB\times AC|$

Area of a Parallelepiped: $A=|AD\bullet(AB\times AC)|$

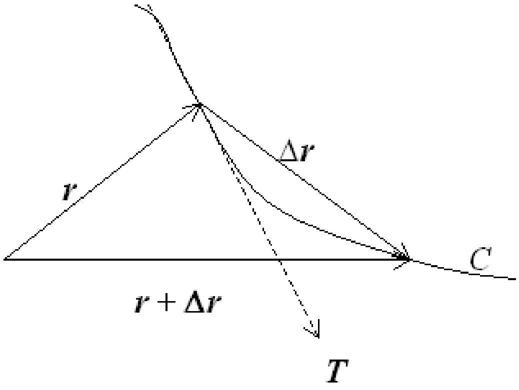
16.6 Space Curves

Where: $r(t)=x(t)i+y(t)j+z(t)k$

Velocity: $v(t) = r'(t) = x'(t)i + y'(t)j + z'(t)k$

Acceleration: $a(t) = v'(t) = r''(t) = x''(t)i + y''(t)j + z''(t)k$

Definition of "s":



The length of the curve from r to r+Δr

Unit Tangent:

$$T = \frac{dr}{ds} = \frac{r'(t)}{|r'(t)|}$$

$$|T| = 1$$

Chain Rule:

$$\frac{dr}{dt} = \frac{dr}{ds} \times \frac{ds}{dt}$$

As $\left|\frac{dr}{ds}\right| = 1$, $\left|\frac{dr}{dt}\right| = \left|\frac{ds}{dt}\right|$=speed

Normal:

$$T \bullet T = 1$$

$$\frac{d}{ds}(T \bullet T) = 0$$

$$\frac{dT}{ds} \bullet T + T \bullet \frac{dT}{ds} = 0$$

$$2T \bullet \frac{dT}{ds} = 0$$

$$T \bullet \frac{dT}{ds} = 0$$

As T is tangent to the curve, $\frac{dT}{ds}$ is normal

$$N = \frac{\left(\frac{dT}{ds}\right)}{\left|\frac{dT}{ds}\right|}$$

Curvature:

$$\frac{dT}{ds} = \left|\frac{dT}{ds}\right| N = \kappa N$$

$$\therefore \kappa = \left|\frac{dT}{ds}\right| = \frac{|r'(t) \times r''(t)|}{|r'(t)|^3} = \frac{|v(t) \times a(t)|}{|v(t)|^3}$$

Unit Binomial: $B = T \times N$

Tortion: $\tau = \left|\frac{dB}{ds}\right|$

ABBREVIATIONS

λ = a scalar value
μ = a scalar value
θ = the angle between the vectors
$\underline{a}$ = a vector
$\underline{b}$ = a vector
k = a scalar value
l = a scalar value
$\underline{n}$ = the normal vector
$\underline{r}$ = the resultant vector

PART 17: SERIES

17.1 MISCELLANEOUS

General Form: $S_n = a_1 + a_2 + a_2 + a_4 + ... + a_n = \sum_{n=1}^{n} a_n$

Infinite Form: $S_\infty = a_1 + a_2 + a_2 + a_4 + ... = \sum_{n=1}^{\infty} a_n$

Partial Sum of a Series: $S_i = a_1 + a_2 + a_2 + a_4 + ... + a_i = \sum_{n=1}^{i} a_n$

0.99…=1: $0.\overline{999} = 9\left(\frac{1}{10}\right) + 9\left(\frac{1}{10}\right)^2 + 9\left(\frac{1}{10}\right)^3 + ... = \frac{9\left(\frac{1}{10}\right)}{1 - \frac{1}{10}} = 1$

17.2 TEST FOR CONVERGENCE AND DIVERGENCE

Test For Convergence: $\lim_{n\to\infty}(S_n) = L$, if L exists, it is convergent

Test For Divergence: $\lim_{n\to\infty}(a_n) \neq 0$

Geometric Series

$$\sum_{n=1}^{\infty} ar^{n-1} \begin{cases} Divergent, |r| \geq 1 \\ Convergent, |r| < 1 \end{cases}$$

P Series

$$\sum_{n=1}^{\infty} \frac{1}{x^p} \begin{cases} Divergent, p \leq 1 \\ Convergent, p > 1 \end{cases}$$

The Sandwich Theorem

If there is a positive series so that $a_n \leq b_n \leq c_n$

If $\lim_{n\to\infty}(a_n) = \lim_{n\to\infty}(c_n) = L$, then, $\lim_{n\to\infty}(b_n) = L$

Hence, if a_n & c_n are convergent, b_n must also be convergent

The Integral Test

If $a_n = f_{(x)}$ if $f_{(x)}$ is continuous, positive and decreasing

If S_∞ or $\int_1^\infty f_{(x)}$ is true, then the other is true

Eg:

$$a_n = \frac{1}{n} = f_{(n)} = \frac{1}{x} = f_{(x)}$$

$$\therefore \int_1^\infty f_{(x)} dx = \int_1^\infty \frac{1}{x} dx = [\ln x]_1^\infty = D.N.E.$$

$\therefore a_n$ is divergent

The Direct Comparison Test

If we want to test a_n, and know the behaviour of b_n, where a_n is a series with only non-negative terms

If b_n is convergent and $a_n \le b_n$, then a_n is also convergent

The Limit Comparison Test

If there is a convergent series $\sum_{n=1}^{\infty} c_n$, then if $\lim_{n\to\infty}\left(\frac{a_n}{c_n}\right) < \infty$, then $\sum_{n=1}^{\infty} a_n$ converges

If there is a divergent series $\sum_{n=1}^{\infty} d_n$, then if $\lim_{n\to\infty}\left(\frac{a_n}{d_n}\right) > 0$, then $\sum_{n=1}^{\infty} a_n$ diverges

D'almbert's Ratio Comparison Test

FOR POSITIVE TERMS:

Converges: $\lim_{n\to\infty}\left(\frac{a_{n+1}}{a_n}\right) < 1$

Diverges: $\lim_{n\to\infty}\left(\frac{a_{n+1}}{a_n}\right) > 1$

Not enough information: $\lim_{n\to\infty}\left(\frac{a_{n+1}}{a_n}\right) = 1$

The nth Root Test

For $\sum_{n=1}^{\infty} a_n$ where $a_n \ge 0$, then if $\lim_{n\to\infty}\sqrt[n]{a_n}$,

Converges: $\lim_{n\to\infty}\sqrt[n]{a_n} < 1$

Diverges: $\lim_{n\to\infty}\sqrt[n]{a_n} > 1$

Not enough information: $\lim_{n\to\infty}\sqrt[n]{a_n} = 1$

Abel's Test:

If $\sum_{n=1}^{\infty} a_n$ is positive and decreasing, and $\sum_{n=1}^{\infty} c_n$ is a convergent series.

Then $\sum_{n=1}^{\infty} a_n \times c_n$ converges

Negative Terms

If $\sum_{n=1}^{\infty} |a_n|$ converges, then $\sum_{n=1}^{\infty} a_n$ is said to be absolutely convergent

Alternating Series Test

This is the only test for an alternating series in the form $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} (-1)^n \times b_n$

Let b_n be the sequence of positive numbers. If $b_{n+1} < b_n$ and $\lim_{n\to\infty} b_n = 0$, then the series is convergent.

Alternating Series Error

$|R_n| = |S - s_n| \le b_{n+1}$, where R_n is the error of the partial sum to the n[th] term.

17.3 ARITHMETIC PROGRESSION:

Definition: $a, a+d, a+2d, a+3d, \ldots$

Nth Term: $= a + d(n-1)$

Sum Of The First N Terms: $\sum_{a=1}^{n} a = \frac{n}{2}\left(2a + d(n-1)\right)$

17.4 GEOMETRIC PROGRESSION:

Definition: $a, ar, ar^2, ar^3, \ldots$

Nth Term: $= ar^{n-1}$

Sum Of The First N Terms: $S_n = \sum_{a=1}^{n} a = \frac{a(1-r^n)}{1-r}$

Sum To Infinity: $S_\infty = \lim_{n\to\infty}\left(\frac{a(1-r^n)}{1-r}\right) = \frac{a}{1-r}$ (given $|r| < 1$)

Geometric Mean:

$P, A, Q, \ldots$

$\frac{A}{P} = r, \frac{Q}{A} = r$

$\therefore \frac{A}{P} = \frac{Q}{A} \Rightarrow A^2 = PQ \Rightarrow A = \sqrt{PQ}$

17.5 SUMMATION SERIES

Linear: $1+2+3+4+\ldots$ $\sum_{a=1}^{n} a = \frac{n(n+1)}{2}$

Quadratic: $1^2+2^2+3^2+4^2+\ldots$ $\sum_{a=1}^{n} a^2 = \frac{n(n+1)(2n+1)}{6}$

Cubic: $1^3+2^3+3^3+4^3+\ldots$ $\sum_{a=1}^{n} a^3 = \left(\frac{n(n+1)}{2}\right)^2$

17.6 APPROXIMATION SERIES

Taylor Series

$$f_{(x)} = \sum_{n=0}^{\infty} a_n (x - x_0)^n = \sum_{n=0}^{\infty} \frac{f^{(n)}{}_{(x_0)}}{n!}(x - x_0)^n = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + a_3(x - x_0)^3 + \ldots$$

where, $a_n = \frac{f^{(n)}{}_{(x_0)}}{n!}$

Maclaurun Series

Special case of the Taylor Series where $x_0 = 0$

Linear Approximation:

$$f_{(x)} \approx L_{(x)} = \sum_{n=0}^{1} a_n (x - x_0)^n = \sum_{n=0}^{1} \frac{f^{(n)}{}_{(x_0)}}{n!}(x - x_0)^n = a_0 + a_1(x - x_0)$$

Quadratic Approximation:

$$f_{(x)} \approx Q_{(x)} = \sum_{n=0}^{2} a_n (x - x_0)^n = \sum_{n=0}^{2} \frac{f^{(n)}{}_{(x_0)}}{n!}(x - x_0)^n = a_0 + a_1(x - x_0) + a_2(x - x_0)^2$$

Cubic Approximation:

$$f_{(x)} \approx C_{(x)} = \sum_{n=0}^{3} a_n (x - x_0)^n = \sum_{n=0}^{3} \frac{f^{(n)}{}_{(x_0)}}{n!}(x - x_0)^n = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + a_3(x - x_0)^3$$

17.7 MONOTONE SERIES

Strictly Increasing: $a_{n+1} > a_n$ $\quad \frac{a_{n+1}}{a_n} > 1$

Non-Decreasing: $a_{n+1} \geq a_n$

Strictly Decreasing: $a_{n+1} < a_n$ $\quad \frac{a_{n+1}}{a_n} < 1$

Non-Increasing: $a_{n+1} \leq a_n$

Convergence: A monotone sequence is convergent if it is bounded, and hence the limit exists when $a_n \to \infty$

17.8 RIEMANN ZETA FUNCTION

Form:

$$\zeta(n) = \sum_{k=1}^{\infty} \frac{1}{k^n}$$

$$\zeta(2n) = (-1)^{n+1} \frac{B_{2n}(2\pi)^{2n}}{2(2n)!}$$

Euler's Table:

n=2	$\zeta(2) = \sum_{k=1}^{\infty} \frac{1}{k^2} = 1 + \frac{1}{4} + \frac{1}{9} + ... = \frac{\pi^2}{6}$
n=4	$\zeta(4) = \sum_{k=1}^{\infty} \frac{1}{k^4} = 1 + \frac{1}{16} + \frac{1}{81} + \frac{1}{256} + ... = \frac{\pi^4}{90}$
n=6	$\zeta(6) = \sum_{k=1}^{\infty} \frac{1}{k^6} = 1 + \frac{1}{64} + \frac{1}{729} + \frac{1}{4096} + ... = \frac{\pi^6}{945}$
n=8	$\zeta(8) = \frac{\pi^8}{9450}$

n=10	$\zeta(10)=\dfrac{\pi^{10}}{93555}$
n=12	$\zeta(12)=\dfrac{691\pi^{12}}{638512875}$
n=14	$\zeta(14)=\dfrac{2\pi^{14}}{18243225}$
n=16	$\zeta(16)=\dfrac{3617\pi^{16}}{325641566250}$
n=18	$\zeta(18)=\dfrac{43867\pi^{18}}{38979295480125}$
n=20	$\zeta(20)=\dfrac{174611\pi^{20}}{1531329465290625}$
n=22	$\zeta(22)=\dfrac{155366\pi^{22}}{13447856940643125}$
n=24	$\zeta(24)=\dfrac{236364091\pi^{24}}{201919571963756521875}$
n=26	$\zeta(26)=\dfrac{1315862\pi^{26}}{11094481976030578125}$

Alternating Series:

$$\sum_{n=0}^{\infty}\left(\frac{(-1)^n}{2n+1}\right)^1=\frac{1}{1}-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-\cdots=\arctan 1=\frac{\pi}{4}$$

$$\sum_{n=0}^{\infty}\left(\frac{(-1)^n}{2n+1}\right)^2=\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\cdots=\frac{\pi^2}{8}$$

$$\sum_{n=0}^{\infty}\left(\frac{(-1)^n}{2n+1}\right)^3=\frac{1}{1^3}-\frac{1}{3^3}+\frac{1}{5^3}-\frac{1}{7^3}+\cdots=\frac{\pi^3}{32}$$

$$\sum_{n=0}^{\infty}\left(\frac{(-1)^n}{2n+1}\right)^4=\frac{1}{1^4}+\frac{1}{3^4}+\frac{1}{5^4}+\frac{1}{7^4}+\cdots=\frac{\pi^4}{96}$$

$$\sum_{n=0}^{\infty}\left(\frac{(-1)^n}{2n+1}\right)^5=\frac{1}{1^5}-\frac{1}{3^5}+\frac{1}{5^5}-\frac{1}{7^5}+\cdots=\frac{5\pi^5}{1536}$$

$$\sum_{n=0}^{\infty}\left(\frac{(-1)^n}{2n+1}\right)^6=\frac{1}{1^6}+\frac{1}{3^6}+\frac{1}{5^6}+\frac{1}{7^6}+\cdots=\frac{\pi^6}{960}$$

Proof for n=2:

Taylor Series Expansion: $\sin(x)=x-\dfrac{x^3}{3!}+\dfrac{x^5}{5!}-\dfrac{x^7}{7!}+\ldots$

$$\sin(x) = x(x-\pi)(x+\pi)(x-2\pi)(x+2\pi)...$$

Polynomial Expansion: $$\sin(x) = x(x^2-\pi^2)(x^2-4\pi^2)(x^2-9\pi^2)...$$

$$\sin(x) = Ax\left(1-\frac{x^2}{\pi^2}\right)\left(1-\frac{x^2}{2^2\pi^2}\right)\left(1-\frac{x^2}{3^2\pi^2}\right)...$$

$$\lim_{x\to 0}\left(\frac{\sin(x)}{x}\right) = 1 = A$$

$$x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+... = x\left(1-\frac{x^2}{\pi^2}\right)\left(1-\frac{x^2}{2^2\pi^2}\right)\left(1-\frac{x^2}{3^2\pi^2}\right)..$$

Comparing the Coefficient of x^3:

$$-\frac{1}{3!} = -\frac{1}{\pi^2}-\frac{1}{2^2\pi^2}-\frac{1}{3^2\pi^2}-\frac{1}{4^2\pi^2}...$$

$$\frac{\pi^2}{6} = 1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}...$$

17.9 SUMMATIONS OF POLYNOMIAL EXPRESSIONS

$$\sum_{i=m}^{n} 1 = n+1-m$$

$$\sum_{i=1}^{n} \frac{1}{i} = H_n$$

(Harmonic number)

$$\sum_{i=m}^{n} i = \frac{(n+1-m)(n+m)}{2}$$

$$\sum_{i=0}^{n} i = \sum_{i=1}^{n} i = \frac{n(n+1)}{2}\sum_{i=0}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} = \frac{n^3}{3}+\frac{n^2}{2}+\frac{n}{6}$$

$$\sum_{i=0}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2 = \frac{n^4}{4}+\frac{n^3}{2}+\frac{n^2}{4} = \left[\sum_{i=1}^{n} i\right]^2$$

$$\sum_{i=0}^{n} i^4 = \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30} = \frac{n^5}{5}+\frac{n^4}{2}+\frac{n^3}{3}-\frac{n}{30}$$

$$\sum_{i=0}^{n} i^p = \frac{(n+1)^{p+1}}{p+1}+\sum_{k=1}^{p}\frac{B_k}{p-k+1}\binom{p}{k}(n+1)^{p-k+1}$$ where B_k denotes a Bernoulli number

$$\left(\sum_{i=m}^{n} i\right)^2 = \sum_{i=m}^{n}(i^3-im(m-1))$$

$$\sum_{i=m}^{n} i^3 = \left(\sum_{i=m}^{n} i\right)^2+\sum_{i=m}^{n} i$$

17.10 SUMMATIONS INVOLVING EXPONENTIAL TERMS

Where $x \neq 1$

$$\sum_{i=m}^{n-1} x^i = \frac{x^m - x^n}{1 - x}$$ ($m < n$)

$$\sum_{i=0}^{n-1} x^i = \frac{1 - x^n}{1 - x}$$ (geometric series starting at 1)

$$\sum_{i=0}^{n-1} i x^i = \frac{x - n x^n + (n-1) x^{n+1}}{(1 - x)^2}$$

$$\sum_{i=0}^{n-1} i 2^i = 2 + (n-2) 2^n$$ (special case when $x = 2$)

$$\sum_{i=0}^{n-1} \frac{i}{2^i} = 2 - \frac{n+1}{2^{n-1}}$$ (special case when $x = 1/2$)

$$\sum_{k=0}^{\infty} \frac{z^k}{k!} = e^z$$

$$\sum_{k=0}^{\infty} k \frac{z^k}{k!} = z e^z$$

$$\sum_{k=0}^{\infty} k^2 \frac{z^k}{k!} = (z + z^2) e^z$$

$$\sum_{k=0}^{\infty} k^3 \frac{z^k}{k!} = (z + 3z^2 + z^3) e^z$$

$$\sum_{k=0}^{\infty} k^4 \frac{z^k}{k!} = (z + 7z^2 + 6z^3 + z^4) e^z$$

$$\sum_{k=0}^{\infty} k^n \frac{z^k}{k!} = z \frac{d}{dz} \sum_{k=0}^{\infty} k^{n-1} \frac{z^k}{k!} = e^z T_n(z)$$

where $T_n(z)$ is the Touchard polynomials.

17.11 SUMMATIONS INVOLVING TRIGONOMETRIC TERMS

$$\sum_{k=0}^{\infty} \frac{(-1)^k z^{2k+1}}{(2k+1)!} = \sin z$$

$$\sum_{k=0}^{\infty} \frac{z^{2k+1}}{(2k+1)!} = \sinh z$$

$$\sum_{k=0}^{\infty} \frac{(-1)^k z^{2k}}{(2k)!} = \cos z$$

$$\sum_{k=0}^{\infty} \frac{z^{2k}}{(2k)!} = \cosh z$$

$$\sum_{k=1}^{\infty} \frac{(-1)^{k-1} (2^{2k} - 1) 2^{2k} B_{2k} z^{2k-1}}{(2k)!} = \tan z, |z| < \frac{\pi}{2}$$

$$\sum_{k=1}^{\infty} \frac{(2^{2k}-1)2^{2k}B_{2k}z^{2k-1}}{(2k)!} = \tanh z, |z| < \frac{\pi}{2}$$
$$\sum_{k=0}^{\infty} \frac{(-1)^k 2^{2k}B_{2k}z^{2k-1}}{(2k)!} = \cot z, |z| < \pi$$
$$\sum_{k=0}^{\infty} \frac{2^{2k}B_{2k}z^{2k-1}}{(2k)!} = \coth z, |z| < \pi$$
$$\sum_{k=0}^{\infty} \frac{(-1)^{k-1}(2^{2k}-2)B_{2k}z^{2k-1}}{(2k)!} = \csc z, |z| < \pi$$
$$\sum_{k=0}^{\infty} \frac{-(2^{2k}-2)B_{2k}z^{2k-1}}{(2k)!} = \operatorname{csch} z, |z| < \pi$$
$$\sum_{k=0}^{\infty} \frac{(-1)^k E_{2k}z^{2k}}{(2k)!} = \sec z, |z| < \frac{\pi}{2}$$
$$\sum_{k=0}^{\infty} \frac{E_{2k}z^{2k}}{(2k)!} = \operatorname{sech} z, |z| < \frac{\pi}{2}$$
$$\sum_{k=0}^{\infty} \frac{(2k)!z^{2k+1}}{2^{2k}(k!)^2(2k+1)} = \arcsin z, |z| \le 1$$
$$\sum_{k=0}^{\infty} \frac{(-1)^k(2k)!z^{2k+1}}{2^{2k}(k!)^2(2k+1)} = \operatorname{arsinh} z, |z| \le 1$$
$$\sum_{k=0}^{\infty} \frac{(-1)^k z^{2k+1}}{2k+1} = \arctan z, |z| < 1$$
$$\sum_{k=0}^{\infty} \frac{z^{2k+1}}{2k+1} = \operatorname{artanh} z, |z| < 1$$
$$\ln 2 + \sum_{k=1}^{\infty} \frac{(-1)^{k-1}(2k)!z^{2k}}{2^{2k+1}k(k!)^2} = \ln\left(1+\sqrt{1+z^2}\right), |z| \le 1$$
$$\sum_{k=1}^{\infty} \frac{\sin(k\theta)}{k} = \frac{\pi-\theta}{2}, 0 < \theta < 2\pi$$
$$\sum_{k=1}^{\infty} \frac{\cos(k\theta)}{k} = -\frac{1}{2}\ln(2-2\cos\theta), \theta \in \mathbb{R}$$
$$\sum_{k=0}^{\infty} \frac{\sin[(2k+1)\theta]}{2k+1} = \frac{\pi}{4}, 0 < \theta < \pi$$
$$B_n(x) = -\frac{n!}{2^{n-1}\pi^n}\sum_{k=1}^{\infty}\frac{1}{k^n}\cos\left(2\pi kx - \frac{\pi n}{2}\right), 0 < x < 1$$
$$\sum_{k=0}^{n} \sin(\theta+k\alpha) = \frac{\sin\frac{(n+1)\alpha}{2}\sin(\theta+\frac{n\alpha}{2})}{\sin\frac{\alpha}{2}}$$
$$\sum_{k=1}^{n-1} \sin\frac{\pi k}{n} = \cot\frac{\pi}{2n}$$

$$\sum_{k=0}^{n-1} \csc^2\left(\theta + \frac{\pi k}{n}\right) = n^2 \csc^2(n\theta) \sum_{k=1}^{n-1} \csc^2 \frac{\pi k}{n} = \frac{n^2-1}{3}$$

$$\sum_{k=1}^{n-1} \csc^4 \frac{\pi k}{n} = \frac{n^4+10n^2-11}{45}$$

17.12 INFINITE SUMMATIONS TO PI

$$\pi = \frac{1}{Z} \qquad Z = \sum_{n=0}^{\infty} \frac{((2n)!)^3(42n+5)}{(n!)^6 16^{3n+1}}$$

$$\pi = \frac{4}{Z} \qquad Z = \sum_{n=0}^{\infty} \frac{(-1)^n(4n)!(21460n+1123)}{(n!)^4 441^{2n+1} 2^{10n+1}}$$

$$\pi = \frac{4}{Z} \qquad Z = \sum_{n=0}^{\infty} \frac{(6n+1)\left(\frac{1}{2}\right)_n^3}{4^n(n!)^3}$$

$$\pi = \frac{72}{Z} \qquad Z = \sum_{n=0}^{\infty} \frac{(-1)^n(4n)!(260n+23)}{(n!)^4 4^{4n} 18^{2n}}$$

$$\pi = \frac{3528}{Z} \qquad Z = \sum_{n=0}^{\infty} \frac{(-1)^n(4n)!(21460n+1123)}{(n!)^4 4^{4n} 882^{2n}}$$

17.13 LIMITS INVOLVING TRIGONOMETRIC TERMS

$$\lim_{x\to a} \sin x = \sin a$$

$$\lim_{x\to a} \cos x = \cos a$$

$$\lim_{x\to 0} \frac{\sin x}{x} = 1$$

$$\lim_{x\to 0} \frac{1-\cos x}{x} = 0$$

$$\lim_{x\to 0} \frac{1-\cos x}{x^2} = \frac{1}{2}$$

$$\lim_{x\to n^{\pm}} \tan\left(\pi x + \frac{\pi}{2}\right) = \mp\infty \qquad \text{for any integer } n$$

ABBREVIATIONS

a = the first term
d = A.P. difference
r = G.P. ratio

17.14 POWER SERIES EXPANSION

Exponential:

$$\sum_{k=0}^{\infty} \frac{z^k}{k!} = e^z$$

$$\sum_{k=0}^{\infty} k\frac{z^k}{k!} = ze^z$$
$$\sum_{k=0}^{\infty} k^2\frac{z^k}{k!} = (z+z^2)e^z$$
$$\sum_{k=0}^{\infty} k^3\frac{z^k}{k!} = (z+3z^2+z^3)e^z$$
$$\sum_{k=0}^{\infty} k^4\frac{z^k}{k!} = (z+7z^2+6z^3+z^4)e^z$$
$$\sum_{k=0}^{\infty} k^n\frac{z^k}{k!} = z\frac{d}{dz}\sum_{k=0}^{\infty} k^{n-1}\frac{z^k}{k!} = e^z T_n(z)$$
$$\ln 2 + \sum_{k=1}^{\infty} \frac{(-1)^{k-1}(2k)!z^{2k}}{2^{2k+1}k(k!)^2} = \ln\left(1+\sqrt{1+z^2}\right), |z| \le 1$$

Trigonometric:

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$
$$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!},$$
$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$
$$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}.$$
$$\tan x = \sum_{n=0}^{\infty} \frac{U_{2n+1} x^{2n+1}}{(2n+1)!}$$
$$= \sum_{n=1}^{\infty} \frac{(-1)^{n-1} 2^{2n} (2^{2n}-1) B_{2n} x^{2n-1}}{(2n)!}$$
$$= x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \frac{17}{315}x^7 + \cdots, \qquad \text{for } |x| < \frac{\pi}{2}.$$
$$\csc x = \sum_{n=0}^{\infty} \frac{(-1)^{n+1} 2(2^{2n-1}-1) B_{2n} x^{2n-1}}{(2n)!}$$
$$= x^{-1} + \frac{1}{6}x + \frac{7}{360}x^3 + \frac{31}{15120}x^5 + \cdots, \qquad \text{for } 0 < |x| < \pi.$$
$$\sec x = \sum_{n=0}^{\infty} \frac{U_{2n} x^{2n}}{(2n)!} = \sum_{n=0}^{\infty} \frac{(-1)^n E_{2n} x^{2n}}{(2n)!}$$
$$= 1 + \frac{1}{2}x^2 + \frac{5}{24}x^4 + \frac{61}{720}x^6 + \cdots, \qquad \text{for } |x| < \frac{\pi}{2}.$$

$$\cot x = \sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n} B_{2n} x^{2n-1}}{(2n)!}$$
$$= x^{-1} - \frac{1}{3}x - \frac{1}{45}x^3 - \frac{2}{945}x^5 - \cdots, \qquad \text{for } 0 < |x| < \pi.$$

$$\sum_{k=0}^{\infty} \frac{z^{2k+1}}{(2k+1)!} = \sinh z$$

$$\sum_{k=0}^{\infty} \frac{z^{2k}}{(2k)!} = \cosh z$$

$$\sum_{k=1}^{\infty} \frac{(2^{2k}-1)2^{2k} B_{2k} z^{2k-1}}{(2k)!} = \tanh z, |z| < \frac{\pi}{2}$$

$$\sum_{k=0}^{\infty} \frac{2^{2k} B_{2k} z^{2k-1}}{(2k)!} = \coth z, |z| < \pi$$

$$\sum_{k=0}^{\infty} \frac{-(2^{2k}-2) B_{2k} z^{2k-1}}{(2k)!} = \operatorname{csch} z, |z| < \pi$$

$$\sum_{k=0}^{\infty} \frac{E_{2k} z^{2k}}{(2k)!} = \operatorname{sech} z, |z| < \frac{\pi}{2}$$

$$\sum_{k=0}^{\infty} \frac{(2k)! z^{2k+1}}{2^{2k}(k!)^2(2k+1)} = \arcsin z, |z| \le 1$$

$$\sum_{k=0}^{\infty} \frac{(-1)^k (2k)! z^{2k+1}}{2^{2k}(k!)^2(2k+1)} = \operatorname{arsinh} z, |z| \le 1$$

$$\sum_{k=0}^{\infty} \frac{(-1)^k z^{2k+1}}{2k+1} = \arctan z, |z| < 1$$

$$\sum_{k=0}^{\infty} \frac{z^{2k+1}}{2k+1} = \operatorname{artanh} z, |z| < 1$$

$$\sum_{k=1}^{\infty} \frac{\sin(k\theta)}{k} = \frac{\pi - \theta}{2}, 0 < \theta < 2\pi$$

$$\sum_{k=1}^{\infty} \frac{\cos(k\theta)}{k} = -\frac{1}{2}\ln(2 - 2\cos\theta), \theta \in \mathbb{R}$$

$$\sum_{k=0}^{\infty} \frac{\sin[(2k+1)\theta]}{2k+1} = \frac{\pi}{4}, 0 < \theta < \pi$$

$$B_n(x) = -\frac{n!}{2^{n-1}\pi^n} \sum_{k=1}^{\infty} \frac{1}{k^n} \cos\left(2\pi kx - \frac{\pi n}{2}\right), 0 < x < 1$$

$$\sum_{k=0}^{n} \sin(\theta + k\alpha) = \frac{\sin \frac{(n+1)\alpha}{2} \sin(\theta + \frac{n\alpha}{2})}{\sin \frac{\alpha}{2}}$$

$$\sum_{k=1}^{n-1} \sin \frac{\pi k}{n} = \cot \frac{\pi}{2n}$$

$$\sum_{k=0}^{n-1} \csc^2 \left(\theta + \frac{\pi k}{n}\right) = n^2 \csc^2(n\theta)$$

$$\sum_{k=1}^{n-1} \csc^2 \frac{\pi k}{n} = \frac{n^2 - 1}{3}$$

$$\sum_{k=1}^{n-1} \csc^4 \frac{\pi k}{n} = \frac{n^4 + 10n^2 - 11}{45}$$

$$\sum_{k=0}^{\infty} \frac{2^{2k}(k!)^2}{(k+1)(2k+1)!} z^{2k+2} = (\arcsin z)^2, |z| \le 1$$

Exponential and Logarithm Series:

$$\ln(1+x) = x\left(\frac{1}{1} - x\left(\frac{1}{2} - x\left(\frac{1}{3} - x\left(\frac{1}{4} - x\left(\frac{1}{5} - \cdots\right)\right)\right)\right)\right) \quad \text{for} \quad |x| < 1.$$

$$\ln(x) = \ln\left(\frac{1+y}{1-y}\right) = 2y\left(\frac{1}{1} + \frac{1}{3}y^2 + \frac{1}{5}y^4 + \frac{1}{7}y^6 + \frac{1}{9}y^8 + \cdots\right)$$

$$= 2y\left(\frac{1}{1} + y^2\left(\frac{1}{3} + y^2\left(\frac{1}{5} + y^2\left(\frac{1}{7} + y^2\left(\frac{1}{9} + \cdots\right)\right)\right)\right)\right), \; y = \frac{x-1}{x+1}$$

$$\ln(1+x) = \frac{x^1}{1} - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \cdots = \cfrac{x}{1 - 0x + \cfrac{1^2x}{2 - 1x + \cfrac{2^2x}{3 - 2x + \cfrac{3^2x}{4 - 3x + \cfrac{4^2x}{5 - 4x + \ddots}}}}}$$

$$\ln\left(1+\frac{2x}{y}\right)=\cfrac{2x}{y+\cfrac{x}{1+\cfrac{x}{3y+\cfrac{2x}{1+\cfrac{2x}{5y+\cfrac{3x}{1+\ddots}}}}}}=\cfrac{2x}{y+x-\cfrac{(1x)^2}{3(y+x)-\cfrac{(2x)^2}{5(y+x)-\cfrac{(3x)^2}{7(y+x)}}}}$$

Fourier Series:

$$f_W(x)=\frac{a_0}{2}+\sum_{k=1}^{\infty}a_k\cos(kx)+b_k\sin(kx)$$

$$f_W(x)=\frac{a_0}{2}+a_1\cos(x)+a_2\cos(2x)+\ldots+a_n\cos(nx)+b_1\sin(x)+b_2\sin(2x)+\ldots+b_n\sin(nx)$$

$$a_k=\frac{1}{\pi}\int_0^{2\pi}f(x)\cos(kx)dx \quad k=0,1,2,\ldots,n$$

$$b_k=\frac{1}{\pi}\int_0^{2\pi}f(x)\sin(kx)dx \quad k=1,2,\ldots,n$$

17.15 Bernoulli Expansion:

Fundamentally: $$1^k+2^k+3^k+\ldots+n^k=\begin{cases}\text{A polynomial in n(n+1)} & \text{k odd}\\ \text{(2n+1) x A polynomial in n(n+1)} & \text{k even}\end{cases}$$

Expansions:

$$1+2+3+...+n=\frac{1}{2}n(n+1)$$

$$1+2+3+...+n=\frac{1}{2}n^2+\frac{1}{2}n$$

$$1+2+3+...+n=\frac{1}{2}\left(\binom{2}{0}B_0n^2+\binom{2}{1}B_1n\right)$$

$$1^2+2^2+3^2+...+n^2=(2n+1)\frac{1}{6}n(n+1)$$

$$1^2+2^2+3^2+...+n^2=\frac{1}{3}n^3+\frac{1}{2}n^2+\frac{1}{6}n$$

$$1^2+2^2+3^2+...+n^2=\frac{1}{3}\left(\binom{3}{0}B_0n^3+\binom{3}{1}B_1n^2+\binom{3}{2}B_2n\right)$$

$$1^3+2^3+3^3+...+n^3=(1+2+3+...+n)^2$$

$$1^3+2^3+3^3+...+n^3=\frac{1}{4}(n(n+1))^2$$

$$1^3+2^3+3^3+...+n^3=\frac{1}{4}n^4+\frac{1}{2}n^3+\frac{1}{4}n^2$$

$$1^3+2^3+3^3+...+n^3=\frac{1}{4}\left(\binom{4}{0}B_0n^4+\binom{4}{1}B_1n^3+\binom{4}{2}B_2n^2+\binom{4}{3}B_3n\right)$$

$$1^4+2^4+3^4+...+n^4=(2n+1)\frac{1}{30}n(n+1)(3n(n+1)-1)$$

$$1^4+2^4+3^4+...+n^4=\frac{1}{5}n^5+\frac{1}{2}n^4+\frac{1}{3}n^3-\frac{1}{30}n$$

$$1^4+2^4+3^4+...+n^4=\frac{1}{5}\left(\binom{5}{0}B_0n^5+\binom{5}{1}B_1n^4+\binom{5}{2}B_2n^3+\binom{5}{3}B_3n^2+\binom{5}{4}B_4n\right)$$

$$1^k+2^k+3^k+...+n^k=\frac{1}{k+1}\left(\binom{k+1}{0}B_0n^{k+1}+\binom{k+1}{1}B_1n^{k+1-1}+\binom{k+1}{2}B_2n^{k+1-2}+...+\binom{k+1}{k-1}B_{k-1}n^2+\binom{k+1}{k}B_kn\right)$$

List of Bernoulli Numbers:

n	B(n)
0	1
1	$-\frac{1}{2}$
2	$\frac{1}{6}$
3	0

4	$-\frac{1}{30}$
5	0
6	$\frac{1}{42}$
7	0
8	$-\frac{1}{30}$
9	0
10	$\frac{5}{66}$
11	0
12	$-\frac{691}{2730}$
13	0
14	$\frac{7}{6}$
15	0
16	$-\frac{3617}{510}$
17	0
18	$\frac{43867}{798}$
19	0
20	$-\frac{174611}{330}$

PART 18: ELECTRICAL

18.1 FUNDAMENTAL THEORY

Charge: $q = 6.24 \times 10^{18}$ Coulombs

Current: $I = \dfrac{dq}{dt}$

Resistance: $R = \dfrac{\rho \ell}{A}$

Ohm's Law: $V = IR$

Power: $P = VI = I^2 R = \dfrac{V^2}{R}$

Conservation of Power: $\sum P_{CONSUMED} = \sum P_{DELIVERED}$

Electrical Energy: $W = P \times t = I^2 \times R \times t = \int_0^t P dt$

Kirchoff's Voltage Law: The sum of the volt drops around a close loop is equal to zero.

$$\sum V = 0$$

Kirchoff's Current Law: The sum of the currents entering any junction is equal to the sum of the currents leaving that junction.

$$\sum I_{IN} = \sum I_{OUT}$$

Average Current:

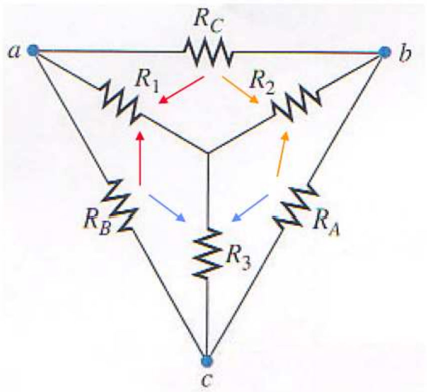
$$I_{AVE} = \frac{1}{T}\int_0^T I(t) dt$$

$$I_{AVE} = \frac{1}{T} \times Area \text{ (under I(t))}$$

RMS Current:

$$\sqrt{\frac{1}{T}\int_0^T (I(t))^2 dt}$$

Δ to Y Conversion:



$$R_1 = \frac{R_B R_C}{R_A + R_B + R_C}$$

$$R_2 = \frac{R_A R_C}{R_A + R_B + R_C}$$

$$R_3 = \frac{R_A R_B}{R_A + R_B + R_C}$$

$$R_A = \frac{R_1R_2 + R_2R_3 + R_1R_3}{R_1}$$

$$R_B = \frac{R_1R_2 + R_2R_3 + R_1R_3}{R_2}$$

$$R_C = \frac{R_1R_2 + R_2R_3 + R_1R_3}{R_3}$$

18.2 COMPONENTS

Resistance in Series: $R_T = R_1 + R_2 + R_3 + \ldots$

Resistance in Parallel: $\frac{1}{R_T} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} + \ldots$

Inductive Impedance: $X_L = j\omega L = j2\pi fL$

Capacitor Impedance: $X_C = -j\frac{1}{\omega C} = -j\frac{1}{2\pi fC}$

Capacitance in Series: $\frac{1}{C_T} = \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3} + \ldots$

Capacitance in Parallel: $C_T = C_1 + C_2 + C_3 + \ldots$

Voltage, Current & Power Summary:

Relation	Resistor (R)	Capacitor (C)	Inductor (L)
v-i:	$v = iR$	$v = \frac{1}{C}\int_{t_0}^{t} i\,dt + v(t_0)$	$v = L\frac{di}{dt}$
i-v:	$i = v/R$	$i = C\frac{dv}{dt}$	$i = \frac{1}{L}\int_{t_0}^{t} i\,dt + i(t_0)$
p or w:	$p = i^2R = \frac{v^2}{R}$	$w = \frac{1}{2}Cv^2$	$w = \frac{1}{2}Li^2$
Series:	$R_{eq} = R_1 + R_2$	$C_{eq} = \frac{C_1C_2}{C_1 + C_2}$	$L_{eq} = L_1 + L_2$
Parallel:	$R_{eq} = \frac{R_1R_2}{R_1 + R_2}$	$C_{eq} = C_1 + C_2$	$L_{eq} = \frac{L_1L_2}{L_1 + L_2}$
At dc:	Same	Open circuit	Short circuit
Circuit variable that cannot change abruptly:	Not applicable	v	i

18.3 THEVENIN'S THEOREM

Thevenin's Theorem:

V_{TH} = Open Circuit Voltage between a & b

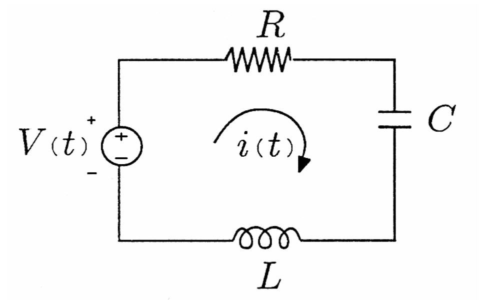
R_{TH} = Short Circuit any voltage source and Open Circuit any current source and calculate R_{TH} as the resistance from a & b. With dependant sources, SC terminals a & b and calculate the current in the wire (I_{SC}). $R_{TH} = \frac{V_{TH}}{I_{SC}}$

Maximum Power Transfer Theorem: $P_{MAX} = \frac{(V_{TH})^2}{4R_{TH}}$, where $R_L = R_{TH}$

18.4 FIRST ORDER RC CIRCUIT

18.5 FIRST ORDER RL CIRCUIT

18.6 SECOND ORDER RLC SERIES CIRCUIT



Calculation using KVL:

$$-V_S + V_R + V_L + V_C = 0$$

$$V_R + V_L + V_C = V_S$$

$$Ri + L\frac{di}{di} + V_C = V_S$$

Circuit current:

$$i = i_C = C\frac{dV_C}{dt}$$

$$\therefore \frac{di}{dt} = C\frac{d^2V_C}{dt^2}$$

$$\therefore RC\frac{dV_C}{dt} + LC\frac{d^2V_C}{dt^2} + V_C = V_S$$

$$LC\frac{d^2V_C}{dt^2} + RC\frac{dV_C}{dt} + V_C = V_S$$

$$\frac{d^2V_C}{dt^2} + \frac{R}{L}\frac{dV_C}{dt} + \frac{1}{LC}V_C = \frac{V_S}{LC}$$

Important Variables

Standard Format: $s^2 + 2\alpha s + \omega_0^{\ 2} = 0$

Damping Factor: $\alpha = \frac{1}{2}\left(\frac{R}{L}\right)$

Natural Frequency: $s = \frac{dV_c}{dt}$

Undamped Natural Frequency: $\omega_0 = \sqrt{\left(\frac{1}{LC}\right)}$

Damping Frequency: $\omega_d = \sqrt{\omega_0{}^2 - \alpha^2}$

Mode Delta: $\Delta = \alpha^2 - \omega_0{}^2$

V_C: $V_C(t) = TRANSIENT + FINAL$

Solving:

$$s^2 + \frac{R}{L}s + \frac{1}{LC} = 0$$

$$s = -\alpha \pm \sqrt{\alpha^2 - \omega_0{}^2} = -\alpha \pm \sqrt{\Delta}$$

Mode 1:

If: $\Delta > 0$, then :

$$s = -\alpha \pm \sqrt{\Delta}$$

$$V_C(t) = TRANSIENT + FINAL$$

$$TRANSIENT = Ae^{s_1 t} + Be^{s_2 t}$$

$$FINAL = V_C(\propto) = V_S$$

$$V_C(t) = Ae^{s_1 t} + Be^{s_2 t} + V_S$$

Finding A & B:

$$V_C(0^+) = V_C(0^-) = V_0$$

$$\therefore A + B + V_S = V_0 \rightarrow A + B = V_0 - V_S$$

$$\frac{dV_c}{dt} = As_1 e^{s_1 t} + Bs_2 e^{s_2 t}$$

$$\frac{dV_C(0^+)}{dt} = \frac{i_C(0^+)}{C} = \frac{i_L(0^+)}{C} = \frac{i_L(0^-)}{C} = \frac{I_0}{C} = As_1 + Bs_2$$

$$\therefore \left.\begin{aligned} V_0 - V_S &= A + B \\ \frac{I_0}{C} &= As_1 + Bs_2 \end{aligned}\right\}$$

Mode 2:

If: $\Delta = 0$, then :

$$s = -\alpha$$

$$V_C(t) = TRANSIENT + FINAL$$

$$TRANSIENT = (A + Bt)e^{st} = (A + Bt)e^{-\alpha t}$$

$$FINAL = V_C(\infty) = V_S$$

$$V_C(t) = (A + Bt)e^{-\alpha t} + V_S$$

Finding A & B:

$$V_C(0^+) = V_C(0^-) = V_0$$
$$\therefore A + V_S = V_0 \rightarrow A = V_0 - V_S$$
$$\frac{dV_c}{dt} = (A + Bt)se^{st} + Be^{st}$$
$$\frac{dV_C(0^+)}{dt} = \frac{i_C(0^+)}{C} = \frac{i_L(0^+)}{C} = \frac{i_L(0^-)}{C} = \frac{I_0}{C} = As + B$$
$$\therefore \left.\begin{aligned} V_0 - V_S &= A \\ \frac{I_0}{C} &= As + B \end{aligned}\right\}$$

Mode 3:

If: $\Delta < 0$, and letting $\omega_d = \sqrt{\omega_0{}^2 - \alpha^2}$, then :

$$s = -\alpha \pm j\omega_d$$
$$V_C(t) = TRANSIENT + FINAL$$
$$TRANSIENT = (A\cos(\omega_d t) + B\sin(\omega_d t))e^{-\alpha t}$$
$$FINAL = V_C(\infty) = V_S$$
$$V_C(t) = (A\cos(\omega_d t) + B\sin(\omega_d t))e^{-\alpha t} + V_S$$

Finding A & B:

$$V_C(0^+) = V_C(0^-) = V_0$$
$$\therefore A + V_S = V_0 \rightarrow A = V_0 - V_S$$
$$\frac{dV_c}{dt} = (-A\omega_d \sin(\omega_d t) + B\omega_d \cos(\omega_d t))e^{-\alpha t} - \alpha(A\cos(\omega_d t) + B\sin(\omega_d t))e^{-\alpha t}$$
$$\frac{dV_C(0^+)}{dt} = \frac{i_C(0^+)}{C} = \frac{i_L(0^+)}{C} = \frac{i_L(0^-)}{C} = \frac{I_0}{C} = B\omega_d - \alpha A$$
$$\therefore \left.\begin{aligned} V_0 - V_S &= A \\ \frac{I_0}{C} &= B\omega_d - \alpha A \end{aligned}\right\}$$

Mode 4:

If: $R = 0$, then :

$$\alpha = 0, \omega_d = \omega_0$$
$$s = \pm j\omega_d = \pm j\omega_0$$
$$V_C(t) = TRANSIENT + FINAL$$
$$TRANSIENT = A\cos(\omega_d t) + B\sin(\omega_d t)$$
$$FINAL = V_C(\infty) = V_S$$
$$V_C(t) = A\cos(\omega_d t) + B\sin(\omega_d t) + V_S$$

Finding A & B:

$$V_C(0^+) = V_C(0^-) = V_0$$

$$\therefore A + V_S = V_0 \rightarrow A = V_0 - V_S$$

$$\frac{dV_c}{dt} = -A\omega_d \sin(\omega_d t) + B\omega_d \cos(\omega_d t)$$

$$\frac{dV_C(0^+)}{dt} = \frac{i_C(0^+)}{C} = \frac{i_L(0^+)}{C} = \frac{i_L(0^-)}{C} = \frac{I_0}{C} = B\omega_d$$

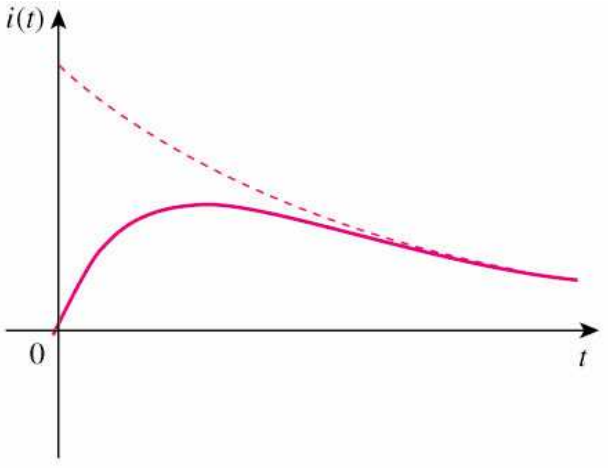
$$\therefore \left.\begin{array}{l} V_0 - V_S = A \\ \frac{I_0}{C} = B\omega_d \end{array}\right\}$$

Current through Inductor:

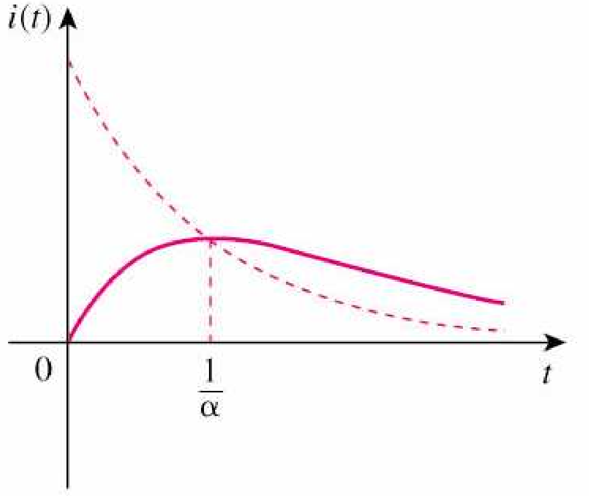
$$i_L = i_C = C\frac{dV_C}{dt}$$

Plotting Modes:

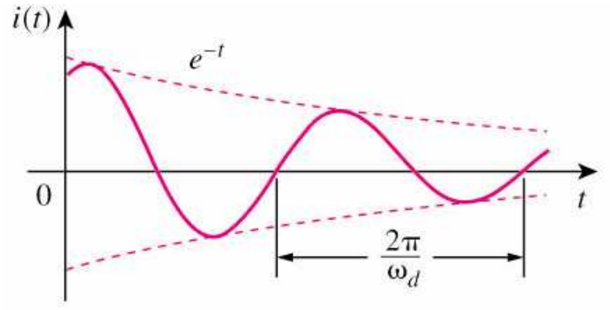
Mode 1: Over Damped



Mode 2: Critically Damped

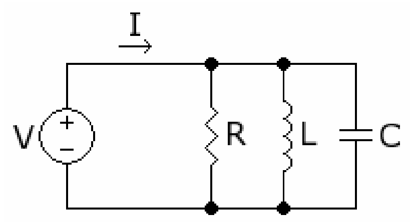


Mode 3: Sinusoidal Damped



Mode 4: Not Damped
(Oscillates indefinitely)

18.7 SECOND ORDER RLC PARALLEL CIRCUIT



Calculation using KCL:

$$i_S = i_R + i_L + i_C$$

$$i_S = \frac{V}{R} + i_L + C\frac{dV}{dt}$$

Node Voltage:

$$V_L = L\frac{di_L}{dt} = V$$

$$\frac{dV}{dt} = L\frac{d^2 i_L}{dt^2}$$

$$\therefore i_S = \frac{L}{R}\frac{di_L}{dt} + i_L + LC\frac{d^2 i_L}{dt^2}$$

$$LC\frac{d^2 i_L}{dt^2} + \frac{L}{R}\frac{di_L}{dt} + i_L = i_S$$

$$\frac{d^2 i_L}{dt^2} + \frac{1}{RC}\frac{di_L}{dt} + \frac{1}{LC}i_L = \frac{1}{LC}i_S$$

Important Variables

Standard Format:	$s^2 + 2\alpha s + \omega_0{}^2 = 0$
Damping Factor:	$\alpha = \frac{1}{2}\left(\frac{1}{RC}\right)$
Undamped Natural Frequency:	$\omega_0 = \sqrt{\left(\frac{1}{LC}\right)}$
Damping Frequency:	$\omega_d = \sqrt{\omega_0{}^2 - \alpha^2}$
Mode Delta:	$\Delta = \alpha^2 - \omega_0{}^2$

Solving:

$$s^2 + \frac{1}{RC}s + \frac{1}{LC} = 0$$

$$s = -\alpha \pm \sqrt{\alpha^2 - {\omega_0}^2} = -\alpha \pm \sqrt{\Delta}$$

18.8 LAPLANCE TRANSFORMATIONS

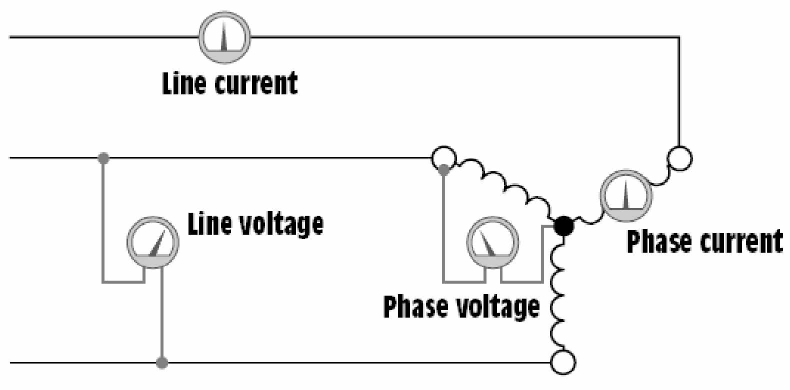
Identities:

1. $\mathcal{L}\{1\} = \frac{1}{s}, \quad s > 0$
2. $\mathcal{L}\{e^{at}\} = \frac{1}{s-a}, \quad s > a$
3. $\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}, \quad s > 0$
4. $\mathcal{L}\{\sin(bt)\} = \frac{b}{s^2+b^2}, \quad s > 0$
5. $\mathcal{L}\{\cos(bt)\} = \frac{s}{s^2+b^2}, \quad s > 0$
6. $\mathcal{L}\{e^{at}t^n\} = \frac{n!}{(s-a)^{n+1}}, \quad s > a$
7. $\mathcal{L}\{e^{at}\sin(bt)\} = \frac{b}{(s-a)^2+b^2}, \quad s > a$
8. $\mathcal{L}\{e^{at}\cos(bt)\} = \frac{s-a}{(s-a)^2+b^2}, \quad s > a$
9. $\mathcal{L}\{f+g\} = \mathcal{L}\{f\} + \mathcal{L}\{g\}$
10. $\mathcal{L}\{cf\} = c\mathcal{L}\{f\}$
11. $\mathcal{L}\{e^{at}f(t)\}(s) = \mathcal{L}\{f\}(s-a)$
12. $\mathcal{L}\{f'\}(s) = s\mathcal{L}\{f\}(s) - f(0)$
13. $\mathcal{L}\{f''\}(s) = s^2\mathcal{L}\{f\}(s) - sf(0) - f'(0)$
14. $\mathcal{L}\left\{f^{(n)}\right\}(s) = s^n\mathcal{L}\{f\}(s) - s^{n-1}f(0) - \ldots - f^{(n-1)}(0)$
15. $\mathcal{L}\{t^n f(t)\}(s) = (-1)^n \frac{d^n}{ds^n}\mathcal{L}\{f\}(s)$

Properties:

Property	$f(t)$	$F(s)$
Linearity	$a_1 f_1(t) + a_2 f_2(t)$	$a_1 F_1(s) + a_2 F_2(s)$
Scaling	$f(at)$	$\frac{1}{a} F\left(\frac{s}{a}\right)$
Time shift	$f(t-a)u(t-a)$	$e^{-as} F(s)$
Frequency shift	$e^{-at} f(t)$	$F(s+a)$
Time differentiation	$\frac{df}{dt}$	$sF(s) - f(0^-)$
	$\frac{d^2 f}{dt^2}$	$s^2 F(s) - sf(0^-) - f'(0^-)$
	$\frac{d^3 f}{dt^3}$	$s^3 F(s) - s^2 f(0^-) - sf'(0^-) - f''(0^-)$
	$\frac{d^n f}{dt^n}$	$s^n F(s) - s^{n-1} f(0^-) - s^{n-2} f'(0^-) - \cdots - f^{(n-1)}(0^-)$
Time integration	$\int_0^t f(t)\, dt$	$\frac{1}{s} F(s)$
Frequency differentiation	$tf(t)$	$-\frac{d}{ds} F(s)$
Frequency integration	$\frac{f(t)}{t}$	$\int_s^\infty F(s)\, ds$
Time periodicity	$f(t) = f(t+nT)$	$\frac{F_1(s)}{1-e^{-sT}}$
Initial value	$f(0^+)$	$\lim_{s\to\infty} sF(s)$
Final value	$f(\infty)$	$\lim_{s\to 0} sF(s)$
Convolution	$f_1(t) * f_1(t)$	$F_1(s) F_2(s)$

18.9 THREE PHASE – Y



Line Voltage: $V_{LINE} = V_{PHASE} \times \sqrt{3}$

Phase Voltage: $V_{PHASE} = \frac{V_{LINE}}{\sqrt{3}}$

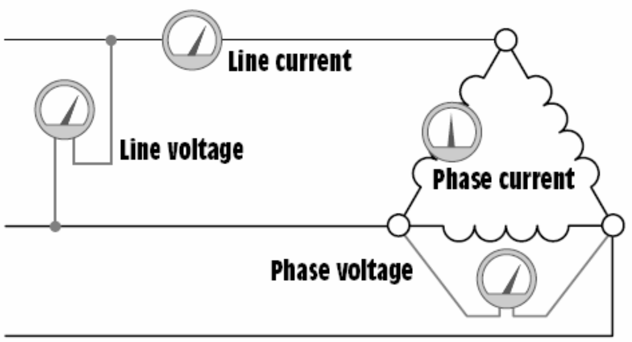
Line Current: $I_{LINE} = I_{PHASE}$

Phase Current: $I_{PHASE} = I_{LINE}$

Power: $S = \sqrt{3} \times V_{LINE} \times I_{LINE}$

$S = 3 \times V_{PHASE} \times I_{PHASE}$

18.10 THREE PHASE – DELTA



Line Voltage: $V_{LINE} = V_{PHASE}$

Phase Voltage: $V_{PHASE} = V_{LINE}$

Line Current: $I_{LINE} = I_{PHASE} \times \sqrt{3}$

Phase Current: $I_{PHASE} = \frac{I_{LINE}}{\sqrt{3}}$

Power: $S = \sqrt{3} \times V_{LINE} \times I_{LINE}$

$S = 3 \times V_{PHASE} \times I_{PHASE}$

18.11 POWER

Instantaneous: $P(t) = V(t) \times I(t)$

Average: $= \frac{1}{T}\int_0^T P(t)dt = \frac{1}{2}V_{MAX}I_{MAX}\cos(\theta_V - \theta_I) = V_{RMS}I_{RMS}\cos(\theta_V - \theta_I)$

Maximum Power: $P_{MAX} = \frac{|V_{TH}|^2}{8R_{TH}}$ where $Z_L = \overline{Z_{TH}}$

Total Power: $= I_{RMS}{}^2 R$

Complex Power:

$$S = V_{RMS}\overline{I_{RMS}}$$

$$S = I_{RMS}{}^2 Z$$

$$S = P + jQ$$

where P = Average or Active Power (W) [positive = load, negative = generator]
where Q = Reactive Power (VAr) [positive = inductive, negative = capacitive]

18.12 Electromagnetics

Definitions:

Magnetic Flux	Φ	Strength of magnetic field	Wb
Reluctance	$\Re$	Relative difficulty for flux to establish	A-t/Wb
Permeability	μ	Relative ease for flux to establish	H/m
Magnetomotive Force	$\Im$	Ability of coil to produce flux	A-t
Flux density	B	Flux per unit area	Wb/m^2 or T
Magnetic Field Intensity	H	MMF per unit length	A-t/m

Permeability of free space: $\mu_0 = 4\pi \times 10^{-7}\, Hm^{-1}$

Magnetic Field Intensity: $H = \frac{\Im}{\ell} = \frac{NI}{\ell}$

Reluctance: $\Re = \frac{1}{\mu A}$

Ohm's Law: $\Phi = \frac{\Im}{\Re}$ OR $\Im = NI$

Magnetic Force on a conductor: $F = BI\ell$

Electromagnetic Induction: $EMF = -N\frac{\Phi_2 - \Phi_1}{t}$

$EMF = Bv\ell$

Magnetic Flux: $\Phi = BA$

Electric Field: $E = \frac{F}{q} = \frac{V}{d}$

Magnetic force on a particle: $F = qvB$

PART 19: GRAPH THEORY

19.1 FUNDAMENTAL EXPLANATIONS:

List of vertices:

$$V = \{v_1, v_2, v_3, ...\}$$

List of edges:

$$E = \{e_1, e_2, e_3, ...\}$$

Subgaphs:

Any subgraph H such that
$V(H) \subset V(G) \,\&\, E(H) \subset E(G)$

Tree:

Any subgraph H where $V(H) = V(G)$, there are no cycles and all verticis are connected.

Degree of vertex:

Number of edges leaving a vertex

$$\sum_{v \in V(G)} d(v) = 2|E(G)|$$

Distance: $d(u,v)$ =Shortest path between u & v

Diameter: $diam(G) = \max_{u \& v \in V(G)} \{d(u,v)\}$

Total Edges in a simple bipartite graph:

$$|E(G)| = \frac{|V(X)||V(Y)|}{2}$$

$$\sum_{x \in X} d(x) = \sum_{y \in Y} d(y)$$

Total Edges in K-regular graph:

$$|E(G)| = \frac{k(k-1)}{2}$$

19.2 FACTORISATION:

1 Factorisation:

A spanning union of 1 Factors and only exists if there are an even number of vertices.

1 Factors of a $K_{n,n}$ bipartite graph:

$$F_1 = [11',22',33',...]$$
$$F_2 = [12',23',34',...]$$
$$F_3 = [13',24',35'...]$$
$$F_n = ...$$

where all numbers are MOD(n)

1 Factors of a K_{2n} graph:

$$F_0 = \{(1,\infty),(2,0),(3,2n-2),...,(n,n+1)\}$$
$$F_i = \{(i,\infty),(i+1,2n-2+1),...,(i+n-1,i+n\}$$
$$F_{2n-2} = ...$$

Where all numbers are MOD(2n-1)

19.3 VERTEX COLOURING:

Chromatic Number:

$\chi(G) \geq 3$ if there are triangles or an odd cycle

$\chi(G) \geq 2$ if is an even cycle

$\chi(G) \geq n$ if is K_n is a subgraph of G

Union/Intersection:

If $G = G_1 \cup G_2$ and $G_1 \cap G_2 = K_m$, then

$$P(G,\lambda) = \frac{P(G_1,\lambda)P(G_2,\lambda)}{P(K_m,\lambda)}$$

Edge Contraction:

$$P(G,\lambda) = P(G-e,\lambda) - P(G.e,\lambda)$$

Common Chromatic Polynomials:

$$P(T_n,\lambda) = \lambda(\lambda-1)^{n-1}$$

$$P(C_n,\lambda) = (\lambda-1)^n + (-1)^n(\lambda-1)$$

$$P(K_n,\lambda) = \lambda(\lambda-1)(\lambda-2)...(\lambda-n+1)$$

- Where the highest power is the number of vertices
- Where the lowest power is the number of components
- Where the coefficient of the 2nd highest power is the number of edges.

19.4 EDGE COLOURING:

Common Chromatic Polynomials:

$$\chi'(G) \geq \Delta(G)$$

$$\chi'(K_{n,n}) = n$$

$$\chi'(C_{2n}) = 2$$

$$\chi'(C_{2n+1}) = 3$$

$$\chi'(K_{2n}) = 2n-1$$

$$\chi'(K_{2n+1}) = 2n+1$$