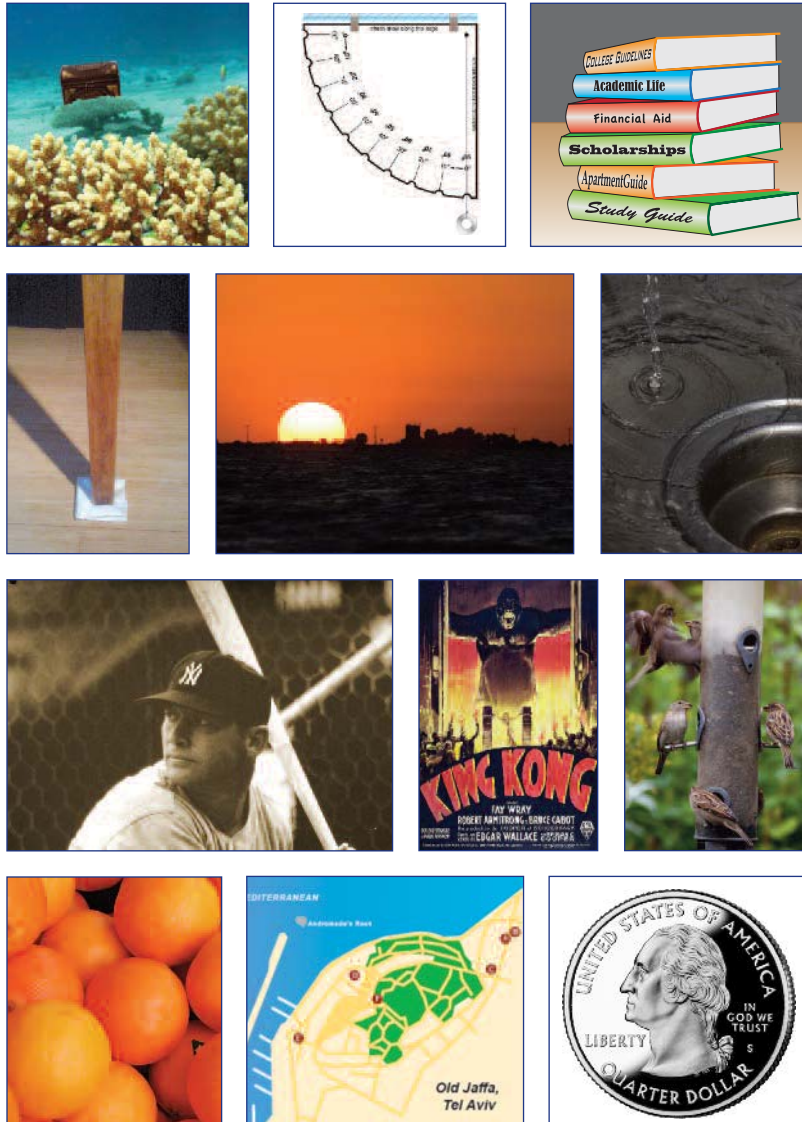
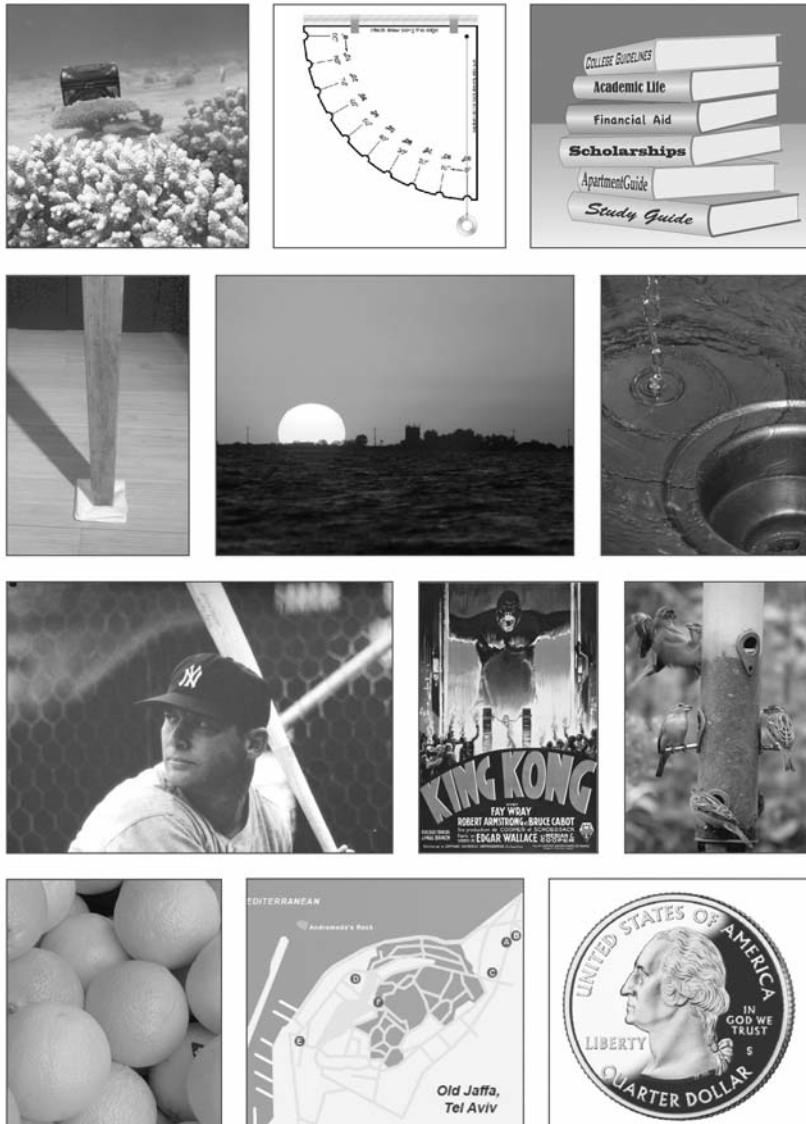


Mathematical Modeling Handbook



Mathematical Modeling Handbook



TEACHERS COLLEGE
COLUMBIA UNIVERSITY

Prepared under the direction of
The Program in Mathematics Education
at
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COMAP

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PREFACE

For most teachers, mathematical modeling represents a new way of “doing” mathematics that makes the addition of modeling activities into instruction seem daunting. This is especially true since modeling, when done properly, requires significant time and effort. In turn, some may be reluctant to include modeling activities into classroom time. It is essential to keep in mind that modeling is one of the eight Standards for Mathematical Practice given in the *Common Core State Standards for Mathematics* (CCSSM) for all grades and is a required conceptual category in high school. Because of this, *modeling cannot be set aside or employed only when spare time arises*. Class time that previously may have been spent using more traditional teaching methods should be converted to time spent on modeling. The integrated nature of mathematical modeling, and in turn the number of curricular standards covered when working through a modeling activity, make modeling activities a very efficient use of class time.

The *Teachers College Mathematical Modeling Handbook* is intended to support the implementation of the CCSSM in the high school mathematical modeling conceptual category. The CCSSM document provides a brief description of mathematical modeling accompanied by 22 star symbols (*) designating modeling standards and standard clusters. The CCSSM approach is to interpret modeling not as a collection of isolated topics but in relation to other standards.

The goal of this *Handbook* is to aid teachers in executing the CCSSM approach by helping students to develop a *mathematical disposition*, that is, to encourage recognition of mathematical opportunities in everyday events. The *Handbook* provides modules and guides for 26 topics together with references to specific CCSSM modeling standards for which the topics may be appropriate. It should be noted that only those standards that have been marked specifically as modeling standards are listed within the modules, however many more standards not marked specifically for modeling are covered.

The *Handbook* begins with an introductory essay by Henry O. Pollak entitled “What is Mathematical Modeling?” Pollak joined the Teachers College faculty in 1987 where he has continued his involvement in modeling and its teaching, emanating from his three decades of work at Bell Laboratories. Pollak has contributed to other COMAP projects and publications including *Mathematics: Modeling Our World* (2000) and “Henry’s Notes” in the newsletter *Consortium*.

Each module is presented in the same format for ease of use. Each contains four sections: (1) Teacher’s Guide – Getting Started, (2) student pages (comprising the student activities), (3) Teacher’s Guide – Possible Solutions, and (4) Teacher’s Guide – Extending the Model.

The first section of each module, “Teacher’s Guide – Getting Started”, is for teachers only. It contains information similar to that in a typical lesson plan: it is meant to give an overview of the module including motivation, the amount of time necessary, what materials and prerequisite knowledge are required, and a general outline describing the student activities in “Worksheet 1” (the first day’s activities) and “Worksheet 2” (the second day’s activities). At the end of this section, the CCSSM modeling standards the module is intended to cover are listed.

The next section of each module consists of the worksheet pages for students. These pages should be photocopied and distributed for student use. It is the teacher’s choice how these lessons should be implemented, but the modules were written with the intention of being a combination of classroom discussion, group, and individual work. The first page of this section is an “artifact page” which lays the foundation for the scenario to be modeled. Occasionally, tables of information, helpful pictures, or tools to be used in the model are included – the so-called “artifacts”. The artifact page concludes with the “leading question” that is the main idea to be addressed and is meant to drive the modeling activities.

The first day's lesson continues after the artifact page and consists of two or three pages. Questions are presented in such a way that students are expected to develop a model to begin to answer the leading question presented on the artifact page. By the end of the first day in most lessons, students craft their model either with mathematics or, sometimes, with actual, physical constructions.

The second day's lesson immediately follows the first day's lesson. It often begins with a "recap" of what happened previously. Definitions sometimes are listed to help drive the model in a mathematical direction in order to focus student attention on specific mathematical ideas. Students continue to work with and refine their model in an effort to answer the leading question more completely or accurately. Sometimes, the lesson proceeds beyond the idea originally posed to help students apply their model to different or more complex scenarios.

Throughout the student pages, there are bracketed notes intended to help guide students through more difficult problems. These are meant to be used if there is trouble moving on from the question and can help you, the teacher, guide the lesson in the direction necessary for completing a model.

The "Teacher's Guide – Possible Solutions" section follows the student pages. Possible answers to the questions posed on both days' student worksheets are given according to the numbered questions. This is intended to give teachers a general guide for how the lesson might progress, but is not meant to be a rigid structure by which classes must abide. Mathematical modeling often can be perceived within several disciplines: students' work should be based on mathematical validity and not on the ability to adhere to the strict mathematical idea presented in the modules.

Each module concludes with a section entitled "Teacher's Guide – Extending the Model" contributed by Pollak. Typically, three kinds of materials for interested and advanced students may be found there: possible extensions of the model developed in the module, other applications of the mathematics of the module, and mathematical extensions of the mathematics within the module. Models are not restricted to one idea and thus have many different uses. "Extending the Model" shows how this is possible.

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Reference – The CCSSM is referenced throughout this *Handbook*, but we will refrain from listing it within each module and only give it here.

National Governors Association Center for Best Practices, Council of Chief State School Officers. (2010). *Common core state standards for mathematics* (CCSSM). Washington, D.C.: National Governors Association Center for Best Practices, Council of Chief State School Officers.

INTRODUCTION

What Is Mathematical Modeling?

Henry O. Pollak
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Mathematical Modeling in a Nutshell

Mathematicians are in the habit of dividing the universe into two parts: mathematics, and everything else, that is, the rest of the world, sometimes called “the real world”. People often tend to see the two as independent from one another – nothing could be further from the truth. When you use mathematics to understand a situation in the real world, and then perhaps use it to take action or even to predict the future, both the real-world situation and the ensuing mathematics are taken seriously. The situations and the questions associated with them may be any size from huge to little. The big ones may lead to lifetime careers for those who study them deeply and special curricula or whole university departments may be set up to prepare people for such careers. Electromagnetic theory, medical imaging, and cryptography are some such examples. At the other end of the scale, there are small situations and corresponding questions, although they may be of great importance to the individuals involved: planning a trip, scheduling the preparation of Thanksgiving dinner, hiring a new assistant, or bidding in an auction. Problems of intelligent citizenship vary greatly in complexity: deciding whether to vote sincerely in the first round of an election, or to vote so as to try to remove the most dangerous threat to your actual favorite candidate; planning the one-way traffic patterns for your downtown; thinking seriously, when the school system argues about testing athletes for steroids, whether you prefer a test that catches almost all the users at the price of designating some non-users as (false) positives, or a test in which almost everybody it catches *is* a user, but misses some of the actual users.

Whether the problem is huge or little, the process of “interaction” between the mathematics and the real world is the same: the real situation usually has so many facets that you can’t take everything into account, so you decide which aspects are most important and keep those. At this point, you have an idealized version of the real-world situation, which you can then translate into mathematical terms. What do you have now? A *mathematical model* of the idealized question. You then apply your mathematical instincts and knowledge to the model, and gain interesting insights, examples, approximations, theorems, and algorithms. You translate all this back into the real-world situation, and you hope to have a theory for the idealized question. But you have to check back: are the results practical, the answers reasonable, the consequences acceptable? If so, great! If not, take another look at the choices you made at the beginning, and try again. This entire process is what is called *mathematical modeling*.

You may be wondering how mathematical modeling differs from what you already teach, particularly, “problem solving”. Problem solving may not refer to the outside world at all. Even when it does, problem solving usually begins with the idealized real-world situation in mathematical terms, and ends with a mathematical result. Mathematical modeling, on the other hand, begins in the “unedited” real world, requires problem *formulating* before problem *solving*, and once the problem is solved, moves back into the real world where the results are considered in their original context. Additionally, it would take us too far afield to discuss *whimsical* problems, where mythical kingdoms and incredible professions and procedures may become the setting of some lovely mathematics. They make no pretense of being problems motivated by the real world.

Mathematical Modeling and Education

Now that we have an idea about what mathematical modeling is in the real world, what do we do about it in mathematics education? One hundred years ago, the big areas – classical physics, astronomy, cartography, and surveying, for instance – were taught in university mathematics departments, perhaps called departments of mathematics and astronomy. Nowadays, in the United States at least, these are taught in science or engineering departments. These branches of science are big and they are very old. What about areas that have become major appliers of mathematics during the last century? Information theory and cryptography may be included in the curricula of electrical engineering, inventory control, programming (as in “linear”), schedul-

ing and queuing in operations research, and fair division and voting in political science. These topics are such exciting new areas of application, often of discrete mathematics, that they frequently have a home in mathematics, as well. Who is going to “own” them in the long run is undecided.

What do we, as mathematicians and mathematics educators, conclude? Many scientific disciplines use mathematics in their development and practice, and when they are faithful to the science they do indeed check which aspects of the situation they have kept and which they have chosen to ignore. Engineers and scientists, be they physical, social, or biological, have not expected mathematics to teach the modeling point of view for them within a scientific framework, although preparing for this kind of reasoning is part of mathematics. Since the scientists will do mathematical modeling anyway, can we just leave mathematical modeling to them? *Absolutely not.* Why not? Mathematics education is at the very least responsible for teaching how to use mathematics in everyday life and in intelligent citizenship, and let’s not forget it. Actually, any separation of science from everyday life is a delusion. Both everyday life and intelligent citizenship often also involve scientific issues. So what really matters in mathematics education is learning and practicing the mathematical modeling process. The particular field of application, whether it is everyday life or being a good citizen or understanding some piece of science, is less important than the experience with this thinking process.

Mathematical Modeling in School

Let us now look at mathematical modeling as an essential component of school mathematics. How successfully have we done this in the past? What are the recollections, and the attitudes, of our graduates? People often say that the mathematics they learned in school and the mathematics that they use in their lives are very different and have little if anything to do with each other. Here’s an example: the textbook or the teacher may have asked how long it takes to drive 20 miles at 40 miles per hour, and accepted the answer of 30 minutes. But how does all this come up in everyday life? When you live 20 miles from the airport, the speed limit is 40 mph, and your cousin is due at 6:00 pm, does that mean you leave at 5:30 pm? Your actual thinking may be quite different. This is rush hour. There are those intersections at which you don’t have the right of way. How long will it take to find a place to park? If you take the back way, the average drive may take longer, but there is much less variability in the total drive time. And don’t forget that the arrival time the airline’s website gives you is the time the plane is expected to touch down on the runway, not when it will start discharging passengers at the gate. And so forth. Contrasting these two thought processes, there is no wonder that students don’t see the connection between mathematics and real life.

We said at the beginning that in mathematical modeling, both the real-world situation and the mathematics are “taken seriously”. What does that mean? It means that the words and images from outside of mathematics are not just idle decorations. It means that the *size* of any numbers involved is realistic, that the *precision* of the numbers is realistic, that the question asked is one that you would ask in the real world. It means that you have considered what aspects of the real-world situation you intend to keep and what aspects you will ignore.

A mathematical model, as we have seen, begins with a situation in the real world which we wish to understand. The particular branch of mathematics that you will end up using may not be known when you start. But then how do you know when and where in the curriculum to discuss a certain modeling problem? If you put it in a section on plane geometry, then students will look for a plane geometric model! Is that what you want? An answer to this difficulty, which is quite real, is that, as in all of mathematics, the learning and the pedagogy are spiral and you return to a major idea many times. In the student’s first experiences with modeling, the particular mathematics to be used will be quite obvious, and that’s fine. Later on, the student may have to consider some alternatives (“Should I try plane geometry, or analytic geometry, or vectors?”), but may very well need help in finding what these alternatives might be, and how to think about the consequences of picking any particular one. At an advanced level, such hints will, we hope, become less necessary.

The Variety of Modules

We have seen that modeling arises in many major disciplines within science, engineering, and even social sciences. As such, it will be at the heart of courses in many disciplines, and at the heart of many varied careers.

Mathematical modeling is also an important aspect of everyday life, where everyone will be better off if they become comfortable with it. It enters many facets of intelligent citizenship. Which kinds of situations do you want to emphasize in school? It is tempting to use modeling as an opportunity to get students thinking about the big issues of our time: world peace, health care, the economy, or the environment. The main point is to develop a favorable disposition and comfort with mathematical modeling, and big issues don't often fit into modules with two lessons.

So, tempting as it may be, the contents of this *Handbook* do not attack any of the major problems of the world. There are *some* modules that can be considered as giving a foretaste of a whole discipline. *A Model Solar System* points towards Newton, Kepler, and the laws of motion and astronomy. Periodic phenomena also appear in both natural and man-made systems, as can be seen in *Sunrise, Sunset. A Bit of Information* gives a taste of the very beginning of information theory, and *State Apportionment* starts some thinking about that particular fair division problem. An introduction to the modeling of epidemics can be associated with *Viral Marketing*.

Both continuous and discrete mathematics are important for modeling. *Bending Steel* and *Water Down the Drain* are examples of continuous problems. An important aspect of a modeling disposition is the ability to make "back-of-the-envelope" estimates that give insight into phenomena that are sometimes surprising. Both *Bending Steel* and the extension to *Water Down the Drain* partake of this aspect of modeling. On the other hand, *A Tour of Jaffa* is discrete, and *Sunken Treasure* has discrete, continuous, and even experimental aspects. And it is sometimes surprising that functions with piecewise definitions occur in the real world as often as they do. Such a problem involves both continuous and discrete thinking. *For the Birds* gives an unexpected example.

Quite naturally, the modules involve a wide variety of high school mathematical topics. Looking for a function with particular properties is at the heart of *A Bit of Information* (logarithms) and of *Rating Systems* (a logistic curve). Another method of looking for a function is to fit a curve to data, which is part of *A Model Solar System*. It is also part of the mathematical modeling process to progress through various areas of mathematics as you become more adept at a particular modeling situation. Thus geometry, algebra, and trigonometry are all part of the development of *Narrow Corridor. Sunken Treasure*, besides using a variety of forms of mathematical reasoning, even suggests using physics in order to do mathematics!

A number of other important mathematical ideas arise in the course of this collection of modeling problems. For example, in connection with several modules involving probability and statistics, the notion of optimal stopping occurs more than once. It is the central idea in *Picking a Painting* and has an important role in *The Whe to Play*. The Intermediate Value Theorem has a crucial role in *Unstable Table*, a delightful everyday-life application of mathematics towards having a comfortable meal in a restaurant. The logistic curve shows up in *Rating Systems* and Voronoi diagrams in *Gauging Rainfall*. Simple everyday-life situations are found in *For the Birds, Estimating Temperatures, and Changing It Up*. We do have one whimsical problem, *Flipping for a Grade*.

A fundamental aspect of mathematical modeling, as is emphasized many times in the *Common Core State Standards for Mathematics*, in the literature on modeling, and in the present work, is the fact that every model downplays certain aspects of reality, which in turn means that the mathematical results eventually have to be checked against reality. This may lead to successive deepening of the models, which shows up particularly in *Narrow Corridor, A Tour of Jaffa, and Unstable Table*. This may be viewed as a new facet of Polya's dictum, "look for a simpler problem".

It is time to bring this introductory essay to a close. We propose two codas, one for mathematicians and one for mathematics educators.

Coda for Mathematicians

We have discussed a number of examples to show the variety of experiences which this collection is intended to encompass. They illustrate situations from everyday life, from citizenship, and from major quantitative disciplines, situations chosen because they lend themselves to brief introductory experiences in mathematical modeling. Don't get the impression that all of this is an unnatural demand on mathematics education. Far from it, it strengthens the affinity between pure mathematics and its applications. The heart of mathematical modeling, as we have seen, is problem formulating before problem solving. So often in mathematics, we say "prove the following theorem" or "solve the following problem". When we start at this point, we are ignoring the fact that finding the theorem or the right problem was a large part of the battle. By emphasizing problem finding, mathematical modeling brings back to mathematics education this aspect of our subject, and greatly reinforces the unity of the total mathematical experience.

Coda for Mathematics Educators

Probably 40 years ago, I was an invited guest at a national summer conference whose purpose was to grade the AP Examinations in Calculus. When I arrived, I found myself in the middle of a debate occasioned by the need to evaluate a particular student's solution of a problem. The problem was to find the volume of a particular solid which was inside a unit three-dimensional cube. The student had set up the relevant integrals correctly, but had made a computational error at the end and came up with an answer in the millions. (He multiplied instead of dividing by some power of 10.) The two sides of the debate had very different ideas about how to allocate the ten possible points. Side 1 argued, "He set everything up correctly, he knew what he was doing, he made a silly numerical error, let's *take off* a point." Side 2 argued, "He must have been sound asleep! How can a solid inside a unit cube have a volume in the millions?! It shows no judgment at all. Let's *give* him a point."

My recollection is that Side 1 won the argument, by a large margin. But now suppose the problem had been set in a mathematical modeling context. Then it would no longer be an argument just from the traditional mathematics point of view. In a mathematical modeling situation, pure mathematics loses some of its sovereignty. The quality of a result is judged not only by the correctness of the mathematics done within the idealized mathematical situation, but also by the success of the confrontation with reality at the end. If the result doesn't make sense in terms of the original situation in the real world, it's not an acceptable solution.

How would you vote?

A NOTE ON TEACHER EDUCATION AND PROFESSIONAL DEVELOPMENT

While this *Handbook* was written with the goal of providing CCSSM-aligned, “ready-made” worksheets for high school teachers to distribute for student use, it also can be adapted easily for use in undergraduate teacher education programs, pre- and in-service training programs, and professional development. The editors would like to recommend some uses of the *Handbook* for those working within these contexts.

In the early years of CCSSM, a large task will be leading all types of teachers to an understanding of exactly what mathematical modeling is. Henry O. Pollak’s introductory essay in this *Handbook* should help begin to forge this understanding. To further its development, future and current teachers should analyze the progression of each of the modules with particular focus on how students are led to think. This progression closely replicates the processes a working mathematician would use to model. Once the thinking process is understood, an understanding of modeling as a whole will begin to blossom.

The modules in the *Handbook* were written to be accessible to most students. Every student is unique and it is reasonable to try to adapt the modules to the needs of different students. Adaptation is another task that can be undertaken in teacher education and professional development programs. Consideration of students’ needs, capabilities, and interests is important and adaptation of the modules in this *Handbook* is encouraged, given that the modeling process – from variable identification to model revision or refinement and reporting the results – is maintained.

A prepared teacher is one who, among other things, anticipates how students will respond to questions and tasks. Another possible task that can be undertaken in teacher education and professional development is trying to anticipate how students will answer the questions posed in the modules, what questions will cause trouble, and how to respond to these. A prepared teacher also will try to determine what to do to help students persevere in developing the model and, if necessary, what extra information can be provided to a student without “giving away” the solution. Determining other mathematically valid types of models besides those presented in the “Possible Solutions” section is also helpful. A task such as this is one all teachers should learn to undertake before teaching a particular lesson.

An interesting teacher education or professional development task would be to determine where the use of these lessons can be taught in an interdisciplinary context. Several modules can be adapted easily for use in science classrooms; some could even be used in the context of social studies, for instance. The act of developing and teaching interdisciplinary lessons using these modules should help both students and teachers understand that a person who is capable of – or at least understands – mathematical modeling is an informed citizen. This is an important lesson to be learned for anyone.

There are various mathematical topics covered within the *Handbook* that may be unfamiliar to teachers as several of them are not frequently taught even in typical undergraduate mathematics courses. This is particularly true of those topics involving discrete mathematics. The topics covered in the *Handbook* all have the “typical” mathematics at their core – number, algebra, geometry, trigonometry, and statistics – but they also frequently involve mathematics not typically seen in high school curricula. It is well-within reason for teacher education and professional development programs to engage in some “content preparation”, such as presentations or short courses on the areas of mathematics that are not typically covered in many teacher preparation curricula that will allow teachers to become more familiar and comfortable with the mathematics employed in the *Handbook*.

A final suggestion for professional development tasks related to the use of this *Handbook* is to determine the best way to assess students, by both formative and summative means. The act of monitoring and evaluating students’ cognitive processes is much more difficult than the act monitoring and evaluating their fluency with

or acquisition of facts and procedures. Mathematical modeling is both a procedure and a cognitive process, so its evaluation is “tricky”. Those engaged in teacher education and professional development programs are encouraged to devise creative and novel ways to assess the modeling activities included in this *Handbook*.

All of the activities listed above would generally be addressed during the course of a lesson study. Practicing teachers might find lesson study to be a valuable professional development activity related to mathematical modeling, and one that can be undertaken without the need to employ outside resources. Lesson study is a common activity in Japanese schools and it involves several teachers working collaboratively on a single lesson or activity in order to understand how to teach it best. While the whole process of lesson study will not be addressed here, it is recommended that teachers work together to develop plans for exactly how to facilitate the teaching of the modeling activities included in this *Handbook*. Making use of one’s colleagues may prove to be the most important and helpful lesson to be learned from professional development activities.

This set of tasks is certainly not exhaustive, nor do we claim that all the suggested tasks are necessary. We do hope, however, that this *Handbook* provides a valuable and enjoyable resource for teacher education and professional development activities related to CCSSM modeling.

COULD KING KONG EXIST?

Teacher's Guide — Getting Started

Diane R. Murray
Manhattanville College

Purpose

When watching the movie *King Kong*, moviegoers are swept away with the idea of a gigantic gorilla capable of running around, climbing with ease, and, most importantly, saving Ann Darrow from harm. But could this animal really exist? In this two-day lesson, students will investigate surface area, volume, and bone strength to determine if his existence is mathematically possible.

Prerequisites

Knowledge of proportions, surface area, and volume formulas are required for this lesson. Unit conversions also are used.

Materials

Required: Scientific calculator.

Suggested: Internet.

Optional: Base-10 1 cm x 1 cm x 1 cm cubes to help demonstrate how volume increases.

Worksheet 1 Guide

The first three pages of the lesson constitute the first day's work. Students will discover that the area of a regular two-dimensional object is proportional to the square of the scale factor. Applying this to three-dimensional objects, students will learn that the surface area is proportional to the square of the scale factor and the volume is proportional to the cube of the scale factor. Using this knowledge, students will apply it to human weight and height measurements.

Worksheet 2 Guide

The fourth and fifth pages of the lesson constitute the second day's work. The students use their knowledge of surface area and volume increases and combine that with information about the strength of bones to discover if King Kong could really exist. Students will finish by searching for information on the largest animals to have existed.

CCSSM Addressed

N-Q.1: Use units as a way to understand problems and to guide the solution of multi-step problems; choose and interpret units consistently in formulas; choose and interpret the scale and the origin in graphs and data displays.

N-Q.2: Define appropriate quantities for the purpose of descriptive modeling.

N-Q.3: Choose a level of accuracy appropriate to limitations on measurement when reporting quantities.

COULD KING KONG EXIST?

Student Name: _____ Date: _____

A human body contains 206 bones! Each bone in your body is designed for a specific function. The longest and strongest bones in your body are the leg bones; the femur, or the thighbone, is the longest and most powerful. It is required to bear your weight and gravitational pressure. The shaft of the bone is shaped like a hollow cylinder, which helps make it so strong. Without the strength of our leg muscles, we would not be able to run, walk, or even stand upright. This also is true for other animals like chimpanzees and orangutans, but what about fictional creatures such as King Kong?



Public Domain Image. Wikipedia Commons

Leading Question

Given the weight and height of King Kong, could he really have existed?

COULD KING KONG EXIST?

Student Name: _____ Date: _____

1. If one square has sides of length 1 cm while a second square has sides of length 2 cm, how are the two areas related?

Draw diagrams to help demonstrate your solution for any of the problems in the lesson.

2. If one rectangle has width of 1 cm and length of 2 cm and the dimensions of a second rectangle are 3 times that of the first, how are the two areas related?
3. If one right triangle has dimensions 3 cm, 4 cm, and 5 cm for the two legs and hypotenuse, respectively, while a second right triangle has 5 times the dimensions of the first, how are the two areas related?
4. Using the previous exercises, write a rule that describes what happens to its area when a “normal” two-dimensional object is scaled by a certain factor.

COULD KING KONG EXIST?

Student Name: _____ Date: _____

5. If a cube has dimensions 1 cm x 1 cm x 1 cm, what is the surface area of the cube? What is the volume of the cube? Find the surface area and volume of a second cube that has double the dimensions of the first cube (i.e., 2 cm x 2 cm x 2 cm). How are the surface areas of the two cubes related? How are the volumes of the two cubes related?

6. What if the original 1 cm x 1 cm x 1 cm cube had its dimensions tripled? How are the surface areas of the original cube and the “tripled” cube related? How are the two volumes related?

7. Create two rules: one that describes what happens to the surface area of a three-dimensional object when its dimensions are increased by a given factor and one that describes what happens to the volume of a three-dimensional object when its dimensions are increased by this same factor.

8. How could the rules that you have created be applied to other shapes? As a person grows in height by a given factor, by what factor might his or her skin grow? By what factor does his or her volume (or weight) grow? If a 10-year-old boy is 51 inches tall and weighs 70.4 pounds and his father is 72 inches tall, determine how much you think the father weighs (assuming that the father and son have similar builds).

9. In the *Austin Powers* series, Dr. Evil created Mini-Me to be one-eighth of his own height. Assuming Dr. Evil is 68 inches tall and weighs 200 pounds, how much should Mini-Me weigh? How tall should he be?

*What does “size”
mean here?*

COULD KING KONG EXIST?

Student Name: _____ Date: _____

14. Using the dimensions from question 12, by what factor has King Kong's surface area actually increased compared to the typical male gorilla?

15. From the previous exercises, could King Kong exist? Why or why not?

16. What about Godzilla? He is portrayed as weighing 50,000 tons and 150 feet tall. If scaled down to 6 feet, how much would he weigh? Is this weight plausible? Why or why not?

17. Using what you know about scaling, weight, and bone strength, find information on large beings that exist or existed, such as elephants, dinosaurs, and whales. What are their heights and weights? How could they exist?

COULD KING KONG EXIST?

Teacher's Guide — Possible Solutions

The solutions shown represent only some possible solution methods. Please evaluate students' solution methods on the basis of mathematical validity.

1. The area of the second square will be 4 times that of the original square.
2. The first rectangle has area of 2 cm^2 while the second rectangle has dimensions $3 \text{ cm} \times 6 \text{ cm}$ and an area of 18 cm^2 . The second area is 9, or 3^2 , times that of the original rectangle.
3. The first triangle has area of 6 cm^2 while the second triangle has dimensions $15 \text{ cm} \times 20 \text{ cm} \times 25 \text{ cm}$ and an area of 150 cm^2 . The second area is 25, or 5^2 , times that of the original triangle.
4. The area of a regular two-dimensional object is proportional to the square of the scale factor.
5. The surface area of the cube is 6 cm^2 . The volume of the cube is 1 cm^3 . The surface area of the new cube is 24 cm^2 . The volume of the new cube is 8 cm^3 . The new surface area is 4 or 2^2 times that of the original cube. The new volume is 8 or 2^3 times that of the original cube.
6. The surface area of the new cube is 54 cm^2 . The volume of the new cube is 27 cm^3 . The new surface area is 9, or 3^2 , times that of the original cube. The new volume is 27, or 3^3 , times that of the original cube.
7. The surface area of a three-dimensional object is proportional to the square of the scale factor. The volume of a three-dimensional object is proportional to the cube of the scale factor.
8. The skin grows proportional to the square of the scale factor of the height growth. The weight grows proportional to the cube of the scale factor of the height growth. The father weighs approximately 198 pounds.
9. Mini-Me should weigh 25 pounds.
10. The father's surface area is larger than the son's surface area by a factor of approximately 2.
11. The cross-sectional area of the leg bone in the father is double that of the son.
12. The movie King Kong should weigh approximately 32,200 pounds.
13. The cross-sectional area of King Kong's bones should be increased by a factor of approximately 86.
14. King Kong's surface area as compared to the typical male gorilla has increased by a factor of approximately 19.5.
15. No, King Kong's bones would crumble from his weight.
16. Godzilla would weigh approximately 6,400 pounds if we were scaled down to a height of 6 feet tall. This is not a plausible weight for an animal that is 6 feet tall.
17. Elephants are the world's largest land animals. Coco, an elephant at the Columbus Zoo, weighed 11,000 pounds and was 10.5 feet tall. The largest dinosaur was the *Amphicoelias*, which weighed 250,000 pounds and was nearly 200 feet long. The largest whale is the blue whale weighing in at 300,000 pounds and 100 feet long. Elephants and dinosaurs have appropriate bones and whales can exist because they live in water.

COULD KING KONG EXIST?

Teacher's Guide — Extending the Model

A very natural extension of this module is to develop a rough understanding of many limitations governing fauna, flora, and structures on our planet. Just to name a few, very far from random, choices:

- How high can a mountain be?
- How high can a tree grow?
- Why can some animals fly and others cannot? What are possible and what are impossible combinations of weight, area of wings, and speed of flight?
- How do animals keep cool – or keep warm?
- How different really are the jumping abilities of people, kangaroos, grasshoppers, and fleas?

These examples are all taken from COMAP's *For All Practical Purposes*, chapter 18 in the eighth edition.

Scaling

The fact that surface area and volume scale differently as length changes is fundamental to the above kinds of models. If you take a cube of side s , then the surface area is $6s^2$, and the volume is s^3 . So surface area divided by volume is $6/s$. Students will gain more of a feeling for this by trying other simple solid figures. Here is an opportunity to become familiar with the five Platonic regular solids. It is of course easiest to do the computations for a cube, but they are not difficult to carry out for the tetrahedron and the octahedron, as well.

The octahedron has a surface consisting of eight equilateral triangles, so if the side length is s , then the surface area is $8\frac{\sqrt{3}}{4}s^2$. The volume is $\frac{\sqrt{2}}{3}s^3$, so the ratio of surface area to volume is $\frac{3\sqrt{6}}{s}$. There are various ways of computing the volume, but since a cross-section in the usual drawing of an octahedron is a square, you can do it by integration if you like.

The tetrahedron's surface consists of four equilateral triangles, rather than eight for the octahedron. Here, the cross-sections are also equilateral triangles. The volume of a tetrahedron of side s is $\frac{\sqrt{2}}{12}s^3$, making the ratio of surface area to volume $\frac{6\sqrt{6}}{s}$.

The other two Platonic solids, the dodecahedron and the icosahedron, may well require some more learning since formulas for regular pentagons, and the trigonometric functions for $\frac{\pi}{5}$, are less familiar.

(Very few students learn that $\sin(18^\circ) = \frac{\sqrt{5}-1}{4}$.)

Comparing with Spheres

One other extension that some students might find interesting is to compare the volumes of the regular solids to the volume of the smallest sphere in which they can be inscribed. Let's look for two points on a unit cube which are furthest apart and call the distance between them the *diameter* of the cube. This will also be the diameter of the smallest sphere to enclose the cube. For a unit cube, the two points furthest apart are at the ends of the longest diagonal, whose distance then is $\sqrt{3}$. So, the volume of the cube is 1, and the volume of the circumscribing sphere is $\frac{4\pi}{3}\left(\frac{\sqrt{3}}{2}\right)^3$. Hence the cube occupies the fraction $\frac{2}{\pi\sqrt{3}}$ of the

sphere, which is about 0.368. That's not very much! For the octahedron, the computation gives a ratio of $\frac{1}{\pi}$, which is about 0.318. The tetrahedron occupies the smallest part of its circumscribing sphere, and the answer is only $\frac{2}{3\pi\sqrt{3}}$, which is about 0.123.

COULD KING KONG EXIST?

Teacher's Guide — Extending the Model

It is worth noting that the diameter of the circumscribing sphere is the distance between two opposite vertices for both the cube and the octahedron, but this is not the case for the tetrahedron. For both the dodecahedron and the icosahedron, the computations of the smallest circumscribing sphere are more formidable, although the diameter is indeed the distance between two vertices. But these solids do a much better job of trying to fill a sphere.

The sphere by itself solves the so-called “isoperimetric problem”, that is, it has the smallest surface area needed for enclosing a given volume. Hence it would require the least amount of material for a container. Nevertheless, as John W. Tukey observed decades ago, nobody makes spherical milk bottles. The best shapes for packaging are a fascinating problem.

Reference

COMAP. (2009). *For All Practical Purposes: Mathematical Literacy in Today's World* (8th ed.). New York: WH Freeman and Company.

A MODEL SOLAR SYSTEM

Teacher's Guide — Getting Started

Diane R. Murray
Manhattanville College

Purpose

In this two-day lesson, students will create several scale models of the Solar System using everyday items. Open with discussing the size of the universe and aim to steer the conversation towards the size of the astronomical bodies. Pose questions that make students think about how large one astronomical body is compared to another. How can they create a model that considers the scale of the bodies?

Prerequisites

An elementary understanding of the Solar System is especially helpful. Students need to be able to use conversions and rates.

Materials

The table below lists diameters and true mean distance of the planets from the Sun. (source: <http://solarsystem.nasa.gov/planets/index.cfm>).

Astronomical Body	Diameter (miles)	True Mean Distance from the Sun (millions of miles)
Sun	864,337	-
Mercury	3,032	36.0
Venus	7,521	67.2
Earth	7,918	93.0
Moon	2,159	N/A
Mars	4,212	141.6
Jupiter	86,881	483.6
Saturn	72,367	886.5
Uranus	31,518	1,783.7
Neptune	30,599	2,795.2

Required: Some of the everyday items listed in the table on the second student page and tools to measure these objects.

Suggested: Access to the internet or other reference source for finding diameters and mean distances, modeling clay is useful for creating spheres with small diameters.

Optional: Spreadsheet software such as Excel, logarithmic graphing paper.

Worksheet 1 Guide

The first three pages of the lesson constitute the first day's work and focus on gathering measurements and the first attempt at devising a model.

Worksheet 2 Guide

The fourth and fifth pages of the lesson constitute the second day's work. Students try two more scales then extend the lesson to mean distance from the Sun.

CCSSM Addressed

N-Q.1, 2, and 3: Reason quantitatively and use units to solve problems.

F-LE.1: Distinguish situations that can be modeled with linear functions.

A MODEL SOLAR SYSTEM

Student Name: _____ Date: _____



Hayden Planetarium, part of the Rose Center for Earth and Space of the American Museum of Natural History in New York City, was redesigned in 2000 to include the “Scales of the Solar System” exhibit, which shows the vast array of sizes of the planets and the Sun. The exhibit demonstrates the massive size of the Solar System by modeling the astronomical bodies as spheres with the Sun being the extremely large sphere partially visible in the top left corner of the photo below. The model Earth, pictured above with the other terrestrial planets, is 10 inches in diameter. How large is the model Sun in the Hayden Planetarium? How large are the other modeled planets? How might you calculate these things?

If you were to build your own model of the Solar System, the first piece of information that you would need to gather would be sizes of the astronomical bodies. One of the easier ways to think about the sizes of the bodies is in terms of diameter. What are the approximate diameters of the Sun, planets, and Moon in our solar system? Use the internet or another reference tool to find these diameters.

Once you have the approximate diameters of the bodies in the Solar System, determine how to model the Solar System physically in the classroom. The table on the following page provides objects and approximate diameters to help create your model.

Astronomical Body	Diameter in miles
Sun	
Mercury	
Venus	
Earth	
Moon	
Mars	
Jupiter	
Saturn	
Uranus	
Neptune	



Photos: ©Hayden Planetarium

Leading Question

What objects found in everyday life might be most helpful in your model? What object would you choose to represent the Earth? Jupiter? The Sun? Are there other objects that you might add?

A MODEL SOLAR SYSTEM

Student Name: _____ Date: _____

Everyday Objects with Approximate Diameters

Possible Objects to Use	Approximate Diameter	Possible Objects to Use	Approximate Diameter
	0.1 inch		
	0.2 inch	Plasma Ball	
	0.3 inch	Hamster Ball	7.5 inches
	0.4 inch	Crystal Ball	8 inches
Marble	0.5 inch	Volleyball	
	0.6 inch	Honeydew Melon	
Tolley Marble	0.75 inch		10 inches
Black Grape	0.8 inch		12 inches
Gnocchi	1 inch	Basketball	
Golf Ball		Beach Ball	20 inches
Racketball Ball		Bean Bag Chair	4 feet
Bouncy Ball	2.5 inches	Wrecking Ball	6 feet
Tennis Ball		Water Walking Ball	6.5 feet
Baseball		Times Square New Year's Eve Ball	
Orange		Tempietto of San Pietro in Rome	15 feet
Bocce Ball	4 inches	Large Cannonball Concretion	18 feet
	5 inches		40 feet
Medium Medicine Ball	6 inches	Epcot Geosphere	165 feet
		1939 New York World's Fair Perisphere	180 feet

1. If the Sun were to be represented by something with a 40-foot diameter, what is the model's scale?
Show your work.

A MODEL SOLAR SYSTEM

Student Name: _____ Date: _____

2. With the scale found in question 1, what everyday object would represent the Earth? The remaining seven planets? The Moon? Show your work.

Model Scale #1: _____				
Planet	True Diameter	Scale Diameter	Everyday Object	Diameter
Sun	864,327 miles			40 feet
Mercury				
Venus				
Earth				
Moon				
Mars				
Jupiter				
Saturn				
Uranus				
Neptune				

3. What flaws does this particular scale have? Is it possible to create this model in your classroom? Why or why not? If not, how might you alter your scale so you could use things you can represent in the classroom?

A MODEL SOLAR SYSTEM

Student Name: _____ Date: _____

Recall from the last class what problems your scale might have had. What might you do differently so that your scale uses objects that you can use in the classroom?

	Model Scale #2: _____			
Planet	True Diameter	Scale Diameter	Everyday Object	Diameter
Sun				
Mercury				
Venus				
Earth				
Moon				
Mars				
Jupiter				
Saturn				
Uranus				
Neptune				

4. How does your new model compare to the first one? Is it smaller or larger in scale? Which aspects of the first model are better than the second? Which aspects of the second are better than the first?

5. Can you create a model that incorporates the best qualities of the first and the best qualities of the second model? Fill in the table on the next page with your new model scale. Compare with your classmates and see if you can find the best possible scale. What qualities should the best scale possess?

A MODEL SOLAR SYSTEM

Student Name: _____ Date: _____

Model Scale #3: _____				
Planet	True Diameter	Scale Diameter	Everyday Object	Diameter
Sun				
Mercury				
Venus				
Earth				
Moon				
Mars				
Jupiter				
Saturn				
Uranus				
Neptune				

Now consider the **mean distance** of each planet from the Sun (exclude the Moon now). Search for the values of the average distance between the planets and the Sun, and see if you can incorporate this into your model. Is the scale also appropriate for your ideal model chosen in the last question?

Model Scale #4: _____				
Planet	True Mean Distance from the Sun (millions of miles)	Scale Mean Distance (miles)	Scaled Mean Distance (feet)	Scaled Mean Distance (inches)
Mercury				
Venus				
Earth				
Mars				
Jupiter				
Saturn				
Uranus				
Neptune				

6. Using this scale, would you be able to see Neptune if you were standing at the Sun? Can you think of a place where you could demonstrate this model?

7. Since the model Earth at Hayden Planetarium is 10 inches in diameter, what is the scale that the designers used? How large are the remaining planets and the Sun?

A MODEL SOLAR SYSTEM

Teacher's Guide — Possible Solutions

Below are three possible scales that your students might use.

	Scale 1: $(1/10^8):1$		Scale 2: $(1/10^9):1$		Scale 3: $[1/(2.5 \times 10^7)]:1$	
Celestial Body	Object	Diameter	Object	Diameter	Object	Diameter
Sun	Hot Air Balloon	40 feet	Bean Bag Chair	4 feet	World's Fair Perisphere	180 feet
Mercury	Golf Ball	1.7 inches	English Pea	0.2 inches	Hamster Ball	7.5 inches
Venus	Bocce Ball	4 inches	Raisin	0.4 inches	Basketball	18 inches
Earth	Grapefruit	5 inches	Marble	0.5 inches	Beach Ball	20 inches
Moon	Gnocchi	1 inch	Nerds candy	0.1 inches	Grapefruit	5 inches
Mars	Bouncy Ball	2.5 inches	Pea	0.3 inches	Small Sugar Pumpkin	10 inches
Jupiter	Wrecking Ball	6 feet	Grapefruit	5 inches	Large Cannonball	18 feet
Saturn	Bean Bag Chair	4 feet	Bocce Ball	4 inches	Tempiette of San Pietro	15 feet
Uranus	Beach Ball	20 inches	Racketball	2.25 inches	Water Walking ball	6.5 feet
Neptune	Basketball	18 inches	Golf Ball	1.7 inches	Wrecking Ball	6 feet

Listed below are objects and diameters that could fill the missing table on the second student page.

Possible Objects to Use	Approximate Diameter	Possible Objects to Use	Approximate Diameter
Nerds Candy	0.1 inch	Grapefruit	5 inches
English Pea	0.2 inch	Plasma Ball	7 inches
Popcorn Kernel	0.3 inch	Volleyball	8.5 inches
Raisin	0.4 inch	Honeydew Melon	9 inches
Acorn	0.6 inch	Small Sugar Pumpkin	10 inches
Golf Ball	1.7 inches	Watermelon	12 inches
Racketball Ball	2.25 inches	Basketball	18 inches
Tennis Ball	2.7 inches	Times Square New Year's Eve Ball	12 feet
Baseball	2.8 inches	First Modern Hot Air Balloon	40 feet
Orange	3 inches	Epcot Geosphere	165 feet

A MODEL SOLAR SYSTEM

Teacher's Guide — Extending the Model

Visualizing the geometry of the planets is an accomplishment. For further work, you may also be interested in looking at the numbers and plotting them and to see how various properties of the planets might be related. The geometry so far has warned us that this will be difficult since the diameters of the four smallest planets and the diameters of the four largest form two clusters that are quite different in diameter. The mean distances from the Sun also span quite a large range and seeing any patterns on a regular piece of graph paper will be difficult.

A mathematical device that makes it easier to see the behavior of numbers spread widely is to plot logarithms of the numbers rather than the numbers themselves. For any set of data that varies over many orders of magnitude, such as the planets, the energies of earthquakes, or the annual incomes of families, plots of the logarithms of the data tend to be very helpful.

When you look at the diameters and the mean distance from the Sun of the various planets and plot them on log-log paper, no pattern becomes immediately evident. There would be a purpose in doing this primarily to obtain yet another set of data about the planets — the time it takes each planet to complete one revolution about the Sun. The unit in which this typically is measured is the time it takes the Earth to do this, namely one Earth year. Take the data for the mean time of revolution of each planet and list them next to the mean distances from the Sun. The sensible thing to do is to plot these on log-log paper. You're able to see one phenomenon right away: the two sets of data move up together.

A closer look at the log-log plot shows that the numbers seem to fall very close to a straight line. This means that for each planet, the logarithm of y , the period of revolution, is linearly related to the logarithm of x , the mean distance from the Sun. The form of the mathematical equation that these data seem to tell you is

$$\log y = a \log x + b$$

where a and b are numbers we can read from the graph.

If you measure the difference in x and y between Mercury and Pluto (when plotted, of course) you should get about 8.2 cm and about 12.3 cm, respectively. The slope of the line is very nearly 1.5 (or $3/2$).

This says that

$$\log y = (3/2) \log x + b$$

or

$$\log y^2 = \log x^3 + 2b$$

which gives

$$y^2 = Bx^3$$

with

$$B = 10^{2b}.$$

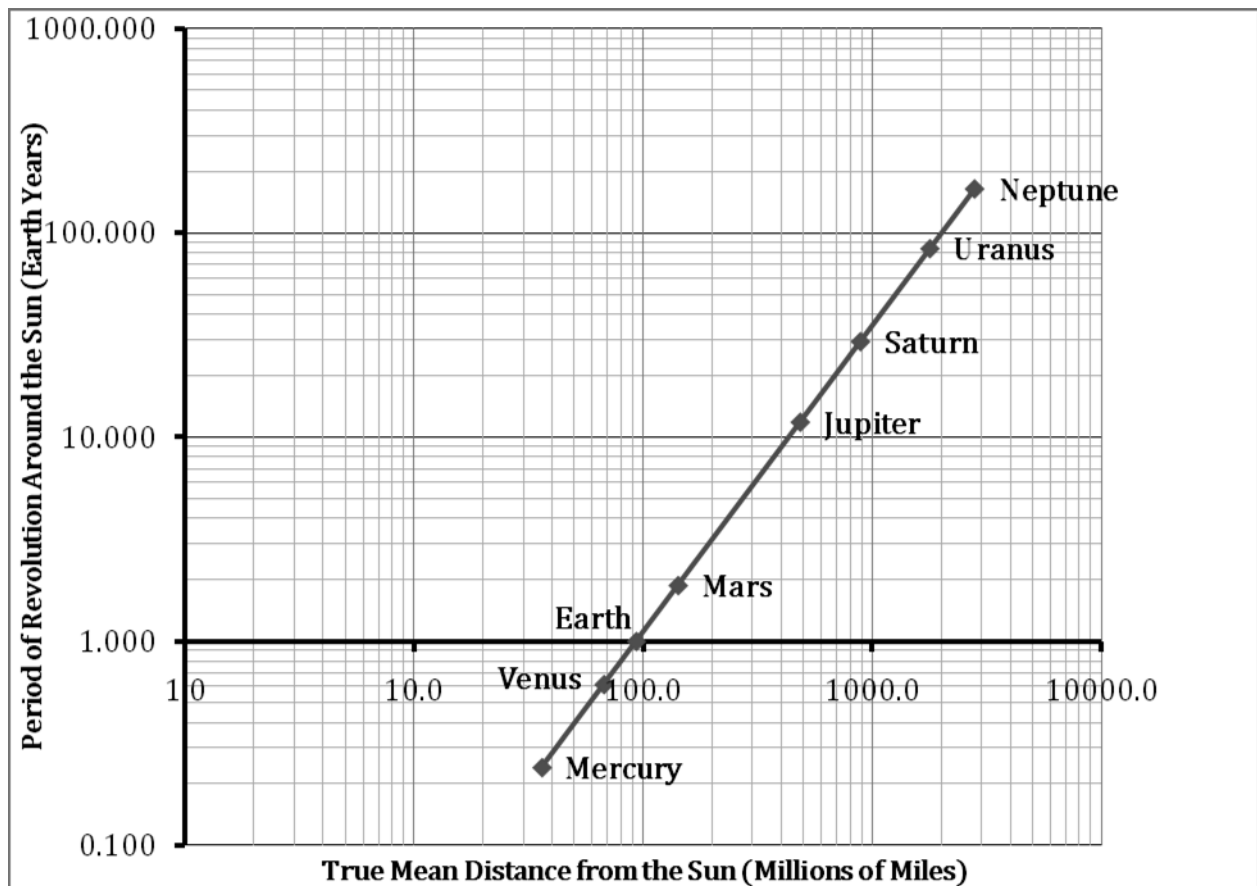
What this shows is Kepler's Third Law — the square of the period of revolution is proportional to the cube of the mean radius of the orbit.

If students are intrigued by logarithmic plots, they may want to investigate the Richter Scale for earthquakes or the loudness of sounds at various distances.

A MODEL SOLAR SYSTEM

Teacher's Guide — Extending the Model

Planet	True Mean Distance from the Sun (millions of miles)	Period of Revolution around the Sun (Earth Years)
Mercury	36.0	0.241
Venus	67.2	0.615
Earth	93.0	1.000
Mars	141.6	1.881
Jupiter	483.6	11.863
Saturn	886.5	29.447
Uranus	1783.7	84.017
Neptune	2795.2	164.791



FOR THE BIRDS

Teacher's Guide — Getting Started

Heather Gould
Stone Ridge, NY

Purpose

In this two-day lesson, students are challenged to consider the different physical factors that affect real-world models. Students are asked to find out how long it will take a birdfeeder — with a constant stream of birds feeding at it — to empty completely.

To begin, explain that the students will be watching over a neighbor's home. This neighbor is an ornithologist (a scientist that studies birds) with a birdfeeder to be looked after. Humans can't come around too often because it will frighten the birds, but they also can't come around too infrequently because the birds will leave if the feeder frequently is empty. The students need to find out when to come back and fill the feeder to ensure that the neighbor and the birds are all happy.

Prerequisites

Students need to be very strong with algebra as there is a heavy reliance on equation manipulation in the lesson.

Materials

Required: (For a physical model) Cardboard box, rice (or sand), cylindrical plastic bottle (a Starbucks Ethos Water bottle, for example), scissors, stopwatches or timers.

Suggested: Graphing paper or a graphing utility.

Optional: None.

Worksheet 1 Guide

The first three pages of the lesson constitute the first day's work. Students are given the opportunity to explore a physical model of a birdfeeder using a cylindrical, plastic bottle as the feeder and rice as the feed. Make sure the bottle is perfectly or very nearly cylindrical. Use scissors to cut "feed holes" (approximately 1 cm in diameter) in the appropriate spots, as indicated in the lesson. Cover the holes so no rice falls out until the experiment is ready to begin (a few students "plugging up" the holes with their fingers is sufficient). Hold the model feeder over a cardboard box so the rice doesn't make a mess. Use stopwatches or other timers to keep track of the total time it takes to empty as well as each of the time periods elapsed at each of the mathematically important moments.

Worksheet 2 Guide

The fourth and fifth pages of the lesson constitute the second day's work. Students need to find out how to model various different situations; they'll learn that each one has a mathematical tie-in to the birdfeeder problems. It turns out that the mathematical model they created for the birdfeeder is sufficient to solve each problem, but this is not obvious until connections are made as to how the problems are related mathematically.

CCSSM Addressed

N-Q.1: Use units as a way to understand problems and to guide the solution of multi-step problems; choose and interpret units consistently in formulas; choose and interpret the scale and the origin in graphs and data displays.

N-Q.2: Define appropriate quantities for the purpose of descriptive modeling.

A-CED.4: Rearrange formulas to highlight a quantity of interest, using the same reasoning as in solving equations.

FOR THE BIRDS

Student Name: _____ Date: _____

Your neighbor, an ornithologist, has to leave for the weekend to do a research study. She has asked you to make sure her birdfeeder always has food in it so that the birds keep coming back throughout the day. Refilling too seldom will cause the birds to look elsewhere for food; refilling too much will scare off the birds.



© Ken Hutchinson | Dreamstime.com

Leading Question

How often should you feed the birds so they keep coming back?

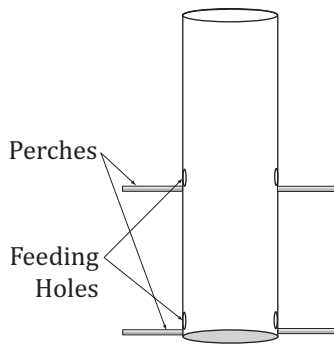
FOR THE BIRDS

Student Name: _____ Date: _____

1. Your neighbor told you that it's important not to fill the feeder too often or to fill the feeder too seldom, so how can you determine how often to fill it?

What's mathematically important about how the birdfeeder empties? Are there any important variables?

2. When you go over first thing in the morning, the birdfeeder — which has 4 holes, one pair near the bottom and another pair about halfway up (shown in the picture) — is nearly full. You check back 45 minutes later and it's about half full. When do you expect it to empty again?

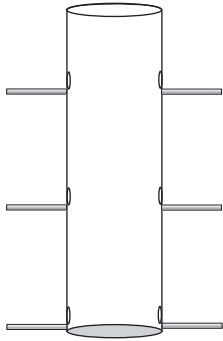


3. You come back 45 minutes later and it's still not nearly empty. Why is that? The birds are still coming by consistently to eat, so they still are hungry. When should you expect the feeder to be nearly empty and ready for you to fill it again?

4. Describe a method for calculating when the birdfeeder should be empty. Use mathematical notation, if you can.

FOR THE BIRDS

Student Name: _____ Date: _____



You did so well taking care of your neighbor's birdfeeder that she recommended you for a weekend job watching over one of her colleague's birdfeeders. This birdfeeder has 6 feeding holes, with pairs equally spaced as shown in the picture.

5. The first morning you get there, you notice that the feeder is about $\frac{2}{3}$ full. You wait a while and notice that it takes about 30 minutes before the feeder is about $\frac{1}{3}$ full. How long will it take before you need to refill the feeder? How long will it take for the feeder to need to be refilled after that?

6. Build a mock birdfeeder like the one above to test your answers from question 5 above. Use a clear, cylindrical container as the birdfeeder and rice as the food. How well did your mathematical model agree with your physical model?

How should you track your findings? Are there certain important events?

7. Write a mathematical description of how to determine how quickly the birdfeeder will empty.

8. Can you *generalize* the description above? Are your answers from questions 4 and 7 similar? How so?

FOR THE BIRDS

Student Name: _____ Date: _____

9. You and 3 of your friends are making crafts for a charity sale. All of you work on Saturday and make 180 in all. On Sunday, only 2 of you can work. How many can you expect to have ready for the sale on Monday morning?
10. There is another charity sale on Saturday. You will make a new type of craft this time. You plan your schedules so that on Monday, 5 of you work; 4 work on Tuesday; 3 work on Wednesday; 2 work on Thursday; and only you make the new craft on Friday. There are 360 crafts done by the end of Tuesday. How many crafts do you expect will be done for the sale?
11. Describe, using words and mathematical notation, how you obtained your answers.
12. Are the birdfeeder problems related to the craft problems? If so, describe the relationship. Is the mathematics involved similar? Why or why not?

FOR THE BIRDS

Student Name: _____ Date: _____

- 13.** You are starting a weekend landscaping business. After the first day, you only finished 25% of the weekend's work. How many friends do you need to hire for tomorrow to help you make sure all the work gets done on time?
- 14.** How is question 13 above similar to the birdfeeder and craft problems? How is it different? What mathematical ideas, if any, are similar? Did you use similar methods?
- 15.** What other types of problems use methods similar to those used above? Make up and solve a problem that uses those methods.
- 16.** What are the types of units used in the problems above? If you know the unit needed in the answer of a problem, can that help you determine how to solve it? Explain.

FOR THE BIRDS

Teacher's Guide — Possible Solutions

The solutions shown represent only some possible solution methods. Please evaluate students' solution methods on the basis of mathematical validity.

1. Important variables to consider are how quickly a portion empties, if birds will always be feeding (the lesson assumes they will, given that they are not frightened by a human tending the feeder too often or frustrated from finding too little food), and how many feeding holes there are and where they're located. The latter two variables often are overlooked.
2. One half of the birdfeeder empties in 45 minutes when the birds are able to access 4 feeding holes. After the halfway point, they are only able to access 2 feeding holes, thereby halving their rate. It takes $45 + 2(45) = 45 + 90 = 135$ minutes = 2 hours, 15 minutes to empty completely. (Often, incorrect answers occur because many people don't consider the different rates.)
3. See answer 2 above.
4. Let F = one feeder, r = the rate at which the feeder empties (the unit is feeders/minute), and t = the time it takes, in minutes. Then $F = rt$ is satisfied if the rate is always constant. The challenge is that the rate changes at the halfway point. So $F = r_1t_1 + r_2t_2$. The initial situation gives $(1/2)F = r_1 \cdot (45)$. Thus, $r_1 = 1/90$. Since the rate slows based on the number of feeding holes available, $r_2 = (1/2)r_1 = (1/2)(1/90) = 1/180$. Then the following is satisfied:
$$1 = (1/90) \cdot 45 + (1/180) \cdot t_2$$
$$1 = (1/2) + (1/180)t_2$$
$$(1/2) = (1/180)t_2$$
$$90 = t_2$$

The birdfeeder empties after $t_1 + t_2$ minutes, which is 135 minutes, or 2 hours and 15 minutes.
5. $F = r_1t_1 + r_2t_2 + r_3t_3$; $t_2 = 30$; $r_2 = 2r_3$; $r_1 = 3r_3$. Also, $(1/3)F = r_2 \cdot (30)$, so $r_2 = 1/90$. Combine these as above to get that $r_3 = 1/180$ and $t_3 = 60$. Finally, $r_1 = 1/60$, $t_1 = 20$. The total time is 110 minutes, or 1 hour and 50 minutes.
6. An accurate physical model will have few differences from the mathematical model.
7. See answer 5 above.
8. See answer 5 above.
9. If 4 people can make 180, then 2 people can make $(2/4)$ as many crafts, or 90. Then the total number of crafts ready by Monday is 270. Mathematically, Crafts = Rate \cdot People. This can be modified as in question 4.
10. There are 9 people each completing a workday Monday and Tuesday and they make a total of 360 crafts. Rearrange the formula to get the rate. Rate = crafts/workdays completed, so rate = $360/9 = 40$ crafts/workday. So by the end of the week, 15 workdays will be completed in all. Thus, crafts = 40 (crafts/workday) \cdot 15 workdays = 600 crafts.
11. See answer 10 above.
12. Both depend heavily on rates.
13. Rate = $(1/4)$ (total job/person). Thus, $(3/4)$ (total job) = $(1/4)$ (total job/person) \cdot 3 people. 3 people are needed.
14. This uses different rates, but all rely heavily on rate issues.
15. Answers will vary. Distance/rate/time problems, $d = rt$, are very common.
16. The unit needed can help with the rearrangement of the necessary formula and can help sort out the "direction" of the problem.

FOR THE BIRDS

Teacher's Guide — Extending the Model

If you plot your data in question 2 to how full the bird feeder is as a function of time, you have three points: at time 0, it is full ($y = 1$); at 45 minutes, it is half full ($y = 1/2 = 0.5$); and your students probably discovered that it would be empty at 135 minutes ($y = 0$). So they have three points: $(0, 1)$; $(45, 0.5)$; and $(135, 0)$.

What do you think happens *between* these points? You expect the birds to eat pretty steadily! So you connect $(0, 1)$ and $(45, 0.5)$ by a straight-line segment, and then $(45, 0.5)$ and $(135, 0)$ also by a straight-line segment. You have a function that is defined *piecewise*. So what would you expect to be the level of the bird feeder to have been at 18 minutes? Probably 0.8. What about at 1 hour and at 2 hours?

Suppose you want the upper part of the feeder to empty in the *same* time as it took the lower part. How can you get it to do that, with the same number of birds involved in each part? One way is to put the upper perches closer to the top! Where should you put them? You should put them $1/3$ of the way down, or you could fail to fill the bird feeder completely when you start. Neither the birds nor the scientists would like that. You can now play with different vertical distances among the rows of perches, and see what variety of patterns you can get.

You have an interesting new question first: when do you think the bird feeder was originally filled? Proceeding as before you will again get a function defined-piecewise, but this time it will consist of *three* pieces. Why?

Something more should be said about piecewise-defined functions. Such functions are seen much more often in modeling the outside world than is generally realized. Here are 3 more examples.

- (i) Post office functions. The simplest example is the postage for a letter as a function of its weight. Highly variable from year-to-year. Other rules, dealing with postage for packages, are more complicated.
- (ii) There was an ad for the price of turkeys at a supermarket the week before Thanksgiving. It said something like 89 cents a pound for birds under 8 pounds, 69 cents a pound between 8 and 14 pounds, and 49 cents a pound above 14 pounds. What could you buy for 7 dollars? 8? 9? In the real world, you may not have all these choices. If you wait too long, you have to settle for whatever size is left.
- (iii) Look at the rpm of an automobile engine as the car starts and accelerates to cruising speed. When you shift from 1st to 2nd, you get onto a different curve and it happens again on the shift from 2nd to high. When shown this function, many students, even those in engineering schools, have trouble understanding what it represents. Jeff Griffiths from Cardiff, Wales was the source of this observation.

Some of these functions are discontinuous, while others have discontinuous first derivatives. They are all defined piecewise, and they all model real situations.

ON SAFARI

Teacher's Guide — Getting Started

Heather Gould & Andrew Sanfratello
Stone Ridge, NY & Mohegan Lake, NY

Purpose

In this two-day lesson, students determine the best way to schedule their time while out on a safari. With only four hours to be out they must use the probabilities of seeing an animal species to determine how much time they should spend there before moving on. The probabilities change with the amount of time spent at a location.

By determining the expected number of animal species seen, students see that the ideal amount of time spent at each location is neither the maximum value nor the minimum value. They justify their conclusion about scheduling using a graph and the slopes of lines from the origin.

Prerequisites

Students should understand weighted average or expected value and they should be able to construct smooth curves through a set of points. Students need to be able to graph points and determine slopes.

Materials

Required: None.

Suggested: Graphing paper.

Optional: None.

Worksheet 1 Guide

The first three pages of the lesson constitute the first day's work. Students use the table of information (given on the first page of the lesson) to determine the best length of time to spend at a location while on safari. The best length is that which will give them the greatest probability of seeing at least one animal of a species per amount of time spent. Thus, students must maximize the unit "probability per minute of seeing at least one animal of a species".

Worksheet 2 Guide

The fourth and fifth pages of the lesson constitute the second day's work. To begin, students are introduced to the idea of a "line segment from the origin". They graph the time spent at a location on the x -axis and the probability of seeing at least one animal of a species on the y -axis and connect the given points with a smooth S-curve. Students then plot the line segments from the origin to the curve to find that the optimal time spent at a location is the one that is associated with the steepest slope from the origin. Finally, they compare and analyze both models.

CCSSM Addressed

N-Q.1: Use units as a way to understand problems and to guide the solution to multi-step problems; choose and interpret units consistently in formulas; choose and interpret the scale and the origin in graphs and data displays.

F-IF.4: For a function that models a relationship between two quantities, interpret key features of graphs and tables in terms of the quantities, and sketch graphs showing key features given a verbal description of the relationship.

S-MD.5: (+) Weigh the possible outcomes of a decision by assigning probabilities to payoff values and finding expected values.

S-MD.7: (+) Analyze decisions and strategies using probability concepts.

ON SAFARI

Student Name: _____ Date: _____

You're going on an African safari! Your goal is to see as many different species of animals as possible. Before you go on your trip, you have to plan how to organize the four hours you will spend in the game park. You will be dropped off and picked up at different locations to spend as much time as you indicate. The field guide (a person who helps tourists on safaris) tells you that the different types of animals are territorial in nature and that seeing two different types of animals at one location is very unlikely. The guide knows where to expect to find the different species, so all you have to do is tell him which species you want to try to see and how long you would like to stay to try to see each one. The guide has provided you with information from last season's safaris (shown below) to help you make your decision. He also tells you that, based on his experience, spending a short amount of time at each location results in fewer sightings because you have to keep quiet and still for a while in order for the animals to feel comfortable enough to make an appearance. Spending too long at a location, however, seems to be a waste because sometimes the animals just don't show up – there's no sense in waiting around for uncooperative animals forever!



Public Domain Image. Wikimedia Commons

Time Spent at Location (in minutes)	Number of Safari Groups	Total Number of Groups that had a Sighting
10	150	6
20	160	24
30	80	24
40	20	72
50	25	90
60	110	88
>60	90	72

Leading Question

How should you plan your time so that you see as many different types of animals as possible?

ON SAFARI

Student Name: _____ Date: _____

1. Examine the table. Which amount of time is best to spend at any location? Which amount of time is worst? Explain your reasoning.

2. If you choose to divide your four total hours equally among several locations, how could you do it?

How many ways can you split up your time equally?

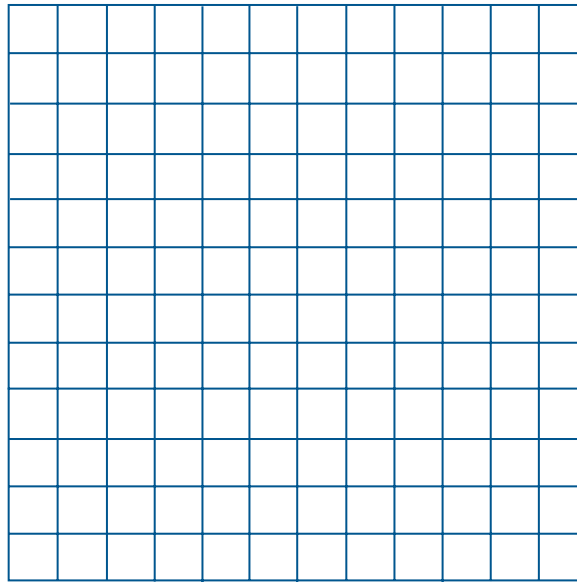
3. In which of the divisions are you most likely to see as many different species as possible? In which of the divisions are you least likely to see many different species? Use a mathematical explanation, if possible.

ON SAFARI

Student Name: _____ Date: _____

A **line segment from the origin** is a line segment with endpoints at the origin $(0, 0)$ and at a point on the curve. A **line that contains a line segment** is the unique line that passes through both of the line segment's endpoints.

8. One way to model the safari schedule is working with a graph of the time spent at any location and the probabilities of sighting at least one animal of a species for those times. Make a graph containing the points $(t, p(t))$ by connecting the points with a smooth curve, where $p(t)$ is the probability of seeing the desired species if you spend t minutes at a location. What features of the graph do you think are important to the model?



9. Sketch line segments from the origin for each of the 10-minute marks from $t = 10$ to $t = 80$ and their associated probabilities. What do you notice about these line segments from the origin? What do you notice about the lines that contain these line segments?

ON SAFARI

Student Name: _____ Date: _____

10. What are the slopes of each of the line segments you drew? What are the meanings of each of these slopes in terms of the model?

What is the unit of measure for the slopes? Does this tell you anything helpful?

11. Is there a point on the curve, different from the ones you have focused on so far, where the line segment from the origin is steeper than the ones you just found? What are the meanings of each of these slopes?

12. Describe a model using line segments from the origin that gives you the greatest chance of seeing many different types of animals. Describe the similarities and differences between your original model and this one. Did you use any of the same mathematical ideas?

13. Do you think the line segment model has any flaws? If so, describe them and suggest a possible solution, if you can.

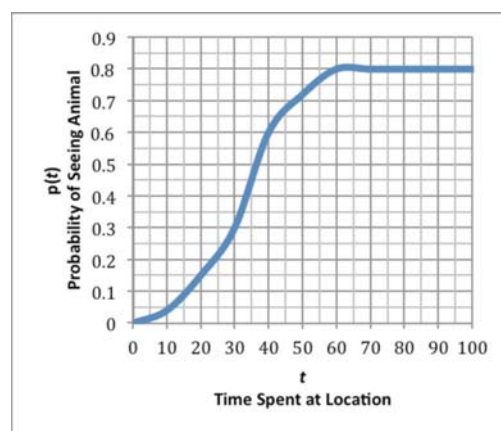
What variables were not considered? Should they have been? What if the optimal time does not divide evenly into the total time?

ON SAFARI

Teacher's Guide — Possible Solutions

The solutions shown represent only some possible solution methods. Please evaluate students' solution methods on the basis of mathematical validity.

1. The probability per minute of spotting at least one animal of a species is greatest at 40 minutes.
2. Answers will vary, but students should note that the 4 total hours allowed can be divided evenly into 10-, 20-, 30-, 40-, and 60-minute intervals.
3. The probability of seeing as many types of animals is greatest when dividing the total time into 40-minute intervals. In a division with only 10-minute intervals, the probability is the least. This can be determined by calculating the expected value per minute.
4. Yes, answers will vary. Unequal divisions of time do not maximize the expected value of animals seen since this is done with the 40-minute interval.
5. In this case, equally divided time provides the best results possible.
6. Answers will vary but should take into account the probability per minute of seeing an animal.
7. Answers will vary but comparing intervals to one another shows that the probability of seeing the animals is highest at the 40-minute interval.
8. Some important features are the inflection points and the point of the S-curve where a line that contains the line segment from the origin always lies at or above the curve itself (See figure to the right. Here, it is at the point (40, 0.6)).
9. Steeper line segments from the origin indicate the greatest probability of seeing at least one animal of a certain species at a location. In this case, the line that contains the line segment with greatest slope always lies at or above the curve while the others do not. (This particular line is a tangent from the origin to the smooth S-curve.)
10. The line segments from the origin have slopes 0.0040, 0.0075, 0.0100, 0.0150, 0.0144, 0.0125, 0.0114, and 0.0100, for 10, 20, 30, 40, 50, 60, 70, and 80 minutes spent at each location, respectively.
11. The 40-minute interval is the best use of time since its line segment from the origin has the steepest slope anywhere on the curve. In fact, the entire S-curve lies below the line that contains the line segment from the origin, and this is not the case anywhere else on the curve. If there were a more efficient use of time, this point would be at the point of the S-curve where a line containing the line segment from the origin always lies at or above the S-curve itself.
12. The model plots the probabilities against the time intervals and uses the line segments from the origin to determine the best way to allot the time. Answers will vary based on each student's original model, but they both should have taken into account the probability per minute of sighting different types of animals. One such model that does this is an expected value model.
13. There are several variables that were not taken into account. Some examples are that the sighting data for different species may differ from the data for all species combined (as given), different species are active during different times of the day, the time to travel between locations was ignored, and sometimes the ideal time spent at each location might not divide evenly into the total time allowed in the game park.



ON SAFARI

Teacher's Guide — Extending the Model

In World War II, the most cogent measure of success in air defense was *attrition*. If a sufficient percentage of attacking bombers were shot down regularly, the enemy could not build new planes and train new crews rapidly enough to keep up their attacks. A basic design principle for defenses against air attacks was to achieve the necessary level of attrition.

It was around 1960 that people realized that this was no longer the correct measure of defense. If each attacker carries a sufficiently powerful warhead, then it takes only one attacker penetrating the defense to destroy the target. The principles for designing defenses needed to be rethought. What follows is the very beginning of the new defense theory, the simplest model of the new reality, a model that provides new insight.

1. There are targets that the attack is trying to destroy and the defense is trying to save. In this simplest model, assume that every target has the same value.
2. The attack has a number of offensive weapons and the basic assumption is that one attacker that penetrates the defense will wipe out the target.
3. The installed defensive equipment is known to the offense. The doctrine by which it will be operated is not.
4. The defense does not know the attack's intended deployment until the battle is actually under way.
5. The defensive weapons will be called "missiles", and assume that targets are sufficiently far apart that missiles installed to defend one target cannot also defend another.
6. In this simplest model, assume that attackers arrive simultaneously at a target so that what matters is the number of defensive missiles that can be launched against a simultaneous attack. This number of defensive missiles will be denoted m .
7. There is a known probability k , $0 < k < 1$, for one missile to destroy the attacker against which it is sent and all missiles succeed independently with the same probability k .

Analysis

If the defense sends two missiles against the same attacker, the attacker escapes one with probability $(1 - k)$ and escapes both with probability $(1 - k)^2$, so that the attacker is destroyed with probability $1 - (1 - k)^2$.

If the defense sends n missiles against one attacker then the attacker is destroyed with probability $A = 1 - (1 - k)^n$.

If the offense sends b attackers against one target, the probability that all attackers are destroyed is A^b , so that the target is lost with probability $1 - A^b$.

You can show easily that the best the defense can do is divide the available defensive missiles as nearly equally as possible among the b attackers. Ignore the fact that equal division may not be exactly possible, and therefore assume that $n = m/b$.

Therefore, if the attack sends b attackers against a target and if the defense has m accessible missiles divided optimally, then the probability that the target is wiped out is

$$P_K = 1 - [1 - (1 - k)^{\frac{m}{b}}]^b .$$

ON SAFARI

Teacher's Guide — Extending the Model

Think of P_K as a function of b . If you plot P_K against b you get an S-curve just like the curve in question 8. The offense's problem of the best choice of b is the same as the problem of picking the viewing time to be spent on each species. Picking too many targets is like making the viewing time too short—you don't have much of a chance to destroy the target/view the species.

Try a numerical example. There are 100 possible targets, a total force of $b = 600$ attackers, and 20 accessible defensive missiles at each target, each of them with $k = 0.5$. If the offense chooses to go after 100 targets, then $b = 6$, $P_K = 0.484$, and the expected number of targets destroyed is 48.4. If the offense attacks only 75 targets, $b = 8$, $P_K = 0.814$, and the expected number of targets destroyed is 61.3. On the other hand, choosing 60 targets will obviously not get the attack as many as 61.3, but "only" 58. So there is an optimum choice for the attack. Where is it? Exactly as in the module, it's at the point where $P_K(b)/b$ is maximized, but that is the slope of the line segment joining the origin to the point $(b, P_K(b))$, and it is maximized if the line segment is actually part of the tangent from the origin to the curve.

There are many important implications of what has been suggested here. For example, the P_K for the optimum b is almost always near 1. This means that any target that the offense chooses to attack is almost certainly lost. So why would the defense even bother to defend it? In order to make the optimum value of b as large as possible. What good is that? It makes the attack so expensive for the offense that they will be not able to attack enough targets to make the overall results worthwhile.

CHOOSING A COLLEGE

Nathan N. Alexander
Charlotte, NC

Teacher's Guide — Getting Started

Purpose

In this two-day lesson, students determine their best-matched college. They use decision-making strategies based on their preferences and ranked choices. This lesson guides students through the process of selecting a list of choices and rating these choices based on their preferences in order to find the college most suited to their preferences and requirements.

Prerequisites

Students must understand how information is sorted in matrices or arrays and they should have experience with problem solving in elementary algebra and utilizing open-ended questioning in mathematics.

Materials

Required: A current issue of *US World & News Report 100 Best Colleges & Universities* (or similar resource).

Suggested: Spreadsheet software (such as MS Excel), internet access.

Optional: None.

Worksheet 1 Guide

The first three pages of the lesson constitute the first day's work in which students generate a list of suitable criteria to help them select a college or university. Students use mathematics to show their preferences of one criterion over another, which may be considered in the model. A set of colleges to consider is determined, and students rate each of the colleges in the set based on how well they meet their preferences for each of the criteria. An initial model for choosing the best school is created.

Worksheet 2 Guide

The fourth and fifth pages of the lesson constitute the second day's work in which students are introduced to and create a decision matrix using the ratings determined on the first day. This becomes a refined model used for determining the best-suited college. Finally, students are introduced to column vectors and use them to weigh each of the important criteria to create a more refined model. The students are not specifically introduced to multiplication of a matrix by a vector, although they are led toward it.

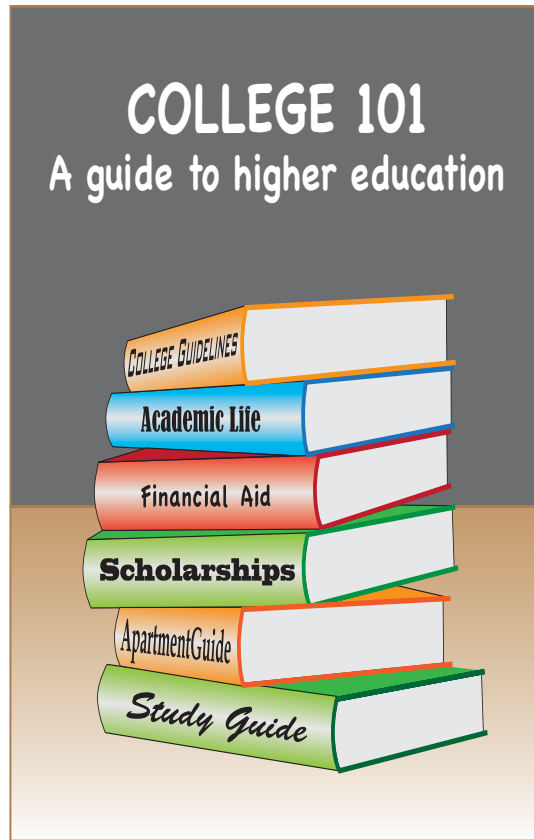
CCSSM Addressed

N-Q.2: Define appropriate quantities for the purpose of descriptive modeling.

CHOOSING A COLLEGE

Student Name: _____ Date: _____

Making decisions can sometimes be quite difficult, especially when it's a decision about where you will spend the next two to four years of your life after you graduate from high school – we're talking about college, of course!



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Leading Question

How can you choose the most suitable college for you?

CHOOSING A COLLEGE

Student Name: _____ Date: _____

1. What criteria are important for you in considering a college or university? Choose 3 – 5 of the criteria that are the most important ones in your opinion.

Some examples of criteria to consider are athletics, academics, costs, financial aid available, and location.

2. Of the criteria you have chosen, which are more important to you? For instance, is tuition more important than location, or is location more important than tuition? List your preferences in order of importance. Explain why each criterion is more important than the next.

What makes one criterion more important to you than another?

3. Choose 3 – 5 colleges in which you are interested and indicate how well they match or meet each of your chosen criteria. Use reference materials such as the *US News & World Report 100 Best Colleges & Universities* or similar resource about colleges to help guide you.

Think about a rating scheme like GPA in which $A = 4$, $B = 3$, $C = 2$, $D = 1$, and $F = 0$.

CHOOSING A COLLEGE

Student Name: _____ Date: _____

4. Use your responses to question 3 to create a model that will help you choose the best school for you.

5. Does your model help you determine which college is best for you? Does it give you your expected results? Does it organize your opinions conveniently? Do you think anyone could use it to help determine their best college choice?

Is there a tie? If so, how does your model help you break it?

6. If your friend has a different list of schools and preferences that he wants to test, how can you use your model to help him? Be specific.

CHOOSING A COLLEGE

Student Name: _____ Date: _____

A **decision matrix** is a tool used to manage a large number of preferences in a simple form. The entries of the decision matrix indicate how well each alternative meets the criterion in question. The rows represent *alternatives* (the objects that are being compared) and the columns represent *criteria* (the characteristics on which the alternatives are being judged). Mathematical operations are used on decision matrices to help reach conclusions about questions related to real-life situations, such as choosing a college.

7. Use your preferences to create a decision matrix for your criteria and the colleges you are considering.

8. Using the decision matrix, how can you determine the final rating of a specific college?

If it looks like College A should be rated higher than College B, what mathematics can you use to show that the individual ratings in each preference “add up” to be more for one than the other?

9. Think of your initial model. If you didn't use a decision matrix, use one to model a method to determine the best college for you. Do the results make sense? If so, how do they make sense? If not, why do you think they do not make sense? Compare your initial model to this new method. If you used a decision matrix model initially, what led you to do so?

CHOOSING A COLLEGE

Student Name: _____ Date: _____

In linear algebra, a **column** (or **row**) **vector** is a matrix consisting only of a single column (or row). Mathematical operations with vectors are used on matrices to help allow for the easy analysis of preference matrices.

10. Look at the relationship between each of your chosen criteria. How can you use mathematics to show that you prefer one criterion over another? Did the decision matrix model you created give equal consideration to all of your criteria? Explain how your model gives either equal or unequal consideration to the criteria and which of these two options should be used in the model.

11. If your model should give unequal consideration to different criteria and it does not, use your responses to question 10 to create a column vector to help you weigh each criteria against one another. The vector, \vec{v} should indicate how you've given unequal consideration to each of your criteria and the i th row should correspond to the i th criterion.

12. Use your decision matrix and column vector to create a modified model and determine the best college for you. What does this model say about the best college for you?

13. Can other real-life decisions be determined using decision matrices? If so, list them and describe briefly how you would go about creating a model for each.

CHOOSING A COLLEGE

Teacher's Guide — Possible Solutions

The solutions shown represent only some possible solution methods. Please evaluate students' solution methods on the basis of mathematical validity.

1. There are various criteria that may be considered. In this answer key we will consider academics, financial aid, and the location of the college as the most important.
2. Here, financial aid is the most important, followed by academics, then location.
3. A scoring or point system can be used in which colleges that meet a given criterion perfectly are given 5 points and colleges that do not meet the criterion at all are given 0 points. So if College I has great financial aid, is a decent school academically, but is a bit far away from home, it will be given the scores 5, 3, and 2 for financial aid, academics, and location, respectively. Similar lines of reasoning gives College II the scores 2, 4, and 5; College III is given the scores 1, 5, and 5; and College IV is given the scores 4, 4, and 0.
4. Each college has been rated on each of the criteria and this information can be summarized in an array, as shown below. The college with the highest sum in its row is the best college choice. Colleges II and III are the best options thus far, but this particular model does not have a method of breaking ties.

	Financial Aid	Academics	Location	Sum
College I	5	3	2	10
College II	2	4	5	11
College III	1	5	5	11
College IV	4	4	0	8

5. The model helps determine better schools, but it has flaws because it cannot help break a tie and determine a single best college. Notice it considers each criterion to have equal value. A person who cares about financial aid much more than academics and location, but still considers those to be the three most important criteria, will not necessarily be satisfied with this model.
6. The same model can be used to help, although different criteria will need to be chosen, his set of colleges to consider will be different, and his ratings will be different. He still has no way to break a tie.
7. The decision matrix associated with these rankings is similar to the array given above. It is shown below.

$$\begin{pmatrix} 5 & 3 & 2 \\ 2 & 4 & 5 \\ 1 & 5 & 5 \\ 4 & 4 & 0 \end{pmatrix}$$

8. The final rating of a college is the sum of the entries in its row. The college with the highest rating is the best one for the student.
9. The decision matrix model and the initial model provided here are very similar, so they have similar benefits and flaws.
10. One way to indicate preference of one criterion over another is to give different point values to them. If academic quality is considered to be about twice as important as location, and financial aid is considered to be a small bit more important than academics, then we can give each criterion a point value. So we can give ratings of 5 points to financial aid, 4 points to academics, and 2 points to location. The initial model gives equal consideration to all criteria, but it probably should not.

CHOOSING A COLLEGE

Teacher's Guide — Possible Solutions

11. Most people won't consider all of their most important criteria all to be equally important, so there should probably be some weighting system in place. The points given in answer 10, namely 5, 4, and 2 points for financial aid, academics, and location, respectively, yield a column matrix whose entries sum to 1. Thus,

$$\vec{v} = \begin{pmatrix} \frac{5}{11} \\ \frac{4}{11} \\ \frac{2}{11} \end{pmatrix} \approx \begin{pmatrix} .45 \\ .36 \\ .18 \end{pmatrix}$$

12. The decision matrix model can be revised by multiplying the decision matrix by the column vector. This will give weights to each of the entries. The product of the multiplication gives College I a rating of 3.69, College II a rating of 3.24, College III a rating of 3.15, and College IV a rating of 3.24. This model gives College I the highest rating because it meets the financial aid criterion the best and it was comparably rated in the initial model. Thus, College I is the best choice using this model. Note that if a student had chosen to give different ranges of point values for different criteria, then this step would unnecessarily inflate ratings. (So if, for example, financial aid ratings were between 0 and 25, academics ratings were between 0 and 20, and location ratings were between 0 and 10, then this step would give too much weight to some criteria and not enough to others.) Thus, it is important for the student to recognize if their initial model or decision matrix model had already taken the relative importance of each criterion into account.
13. This model can be modified to use to help solve many real-life decision problems. One could choose which political candidate is the right one for them to vote for, for instance. In this case, the alternatives would be each candidate and the criteria would be political issues. The candidates would be rated on how well each of their views or voting records matched with the voter in question. The model could be refined to incorporate the importance of each issue to the voter using matrix multiplication by a column vector in which the entries represent the relative importance of each issue to one another for the voter.

CHOOSING A COLLEGE

Teacher's Guide — Extending the Model

Our method for choosing a college came in two steps: in the first, we created a *decision matrix* and rated each of four Colleges (I, II, III, and IV in our example) on each of three criteria (financial aid, academics, and location in our example); in the second, we assigned a weight to each criterion, specifically, 5/11 for financial aid, 4/11 for academics, and 2/11 for location. Notice that these are proportions that add up to 1. In the example, College I was the winner, II and IV were tied for second, and III came in last.

Suppose that instead of (5/11, 4/11, 2/11) as the weights, we had used (x, y, z) with the conditions that each of x , y , and z are greater than or equal to 0, and that the sum of x , y , and z is 1. Can our choice of the four colleges be the winner given that we have appropriately chosen values of x , y , and z ? We can't be sure, but we can find out. Remembering the decision matrix, we see that the score for College I will be $5x + 3y + 2z$. Since it's a lot easier to work with visual representations, and since it's a lot easier to draw pictures in two dimensions than in three, we can try to reduce our work from three dimensions to two. We know that $z = 1 - x - y$. If we make that substitution, we get scores, S , for each of the colleges in just two variables: for College I, the score is $S_I = 5x + 3y + 2(1 - x - y) = 3x + y + 2$; for College II, $S_{II} = -3x - y + 5$; for College III, $S_{III} = -4x + 5$; and for College IV, $S_{IV} = 4x + 4y$.

So College I will win if $S_I = \max(S_I, S_{II}, S_{III}, S_{IV})$. If we want to plot our results, the region of the (x, y) plane in which we look is given by the conditions $x \geq 0$, $y \geq 0$, and also $z = 1 - x - y \geq 0$, which we rewrite as $x + y \leq 1$. Together, these form an isosceles right triangle T in the first quadrant. What will happen is that this triangle will be divided into at most four polygonal regions, and in each of those regions, one of the four colleges will be the winner. In this case, we *do* get four regions, which means that with the right choice of (x, y, z) , any one of the four can be the winner. This will not always happen. Each polygonal region is convex and any segment of each boundary is a segment of a straight line $S_i = S_j$ (where i and j stand for Roman numerals), or else a segment of the three boundary edges of the triangle T . The picture is given on a separate page. A line $S_i = S_j$ divides the plane into two half-planes: in one, $S_i < S_j$, and in the other, $S_i > S_j$. Unless the line happens to go through the origin, a lazy way to tell which is which is to see where the origin should be. (In our example, only the line $S_{II} = S_{III}$ goes through the origin.) This makes each of the four polygonal regions the intersection of half-planes determined by its boundary segments and the boundaries of T .

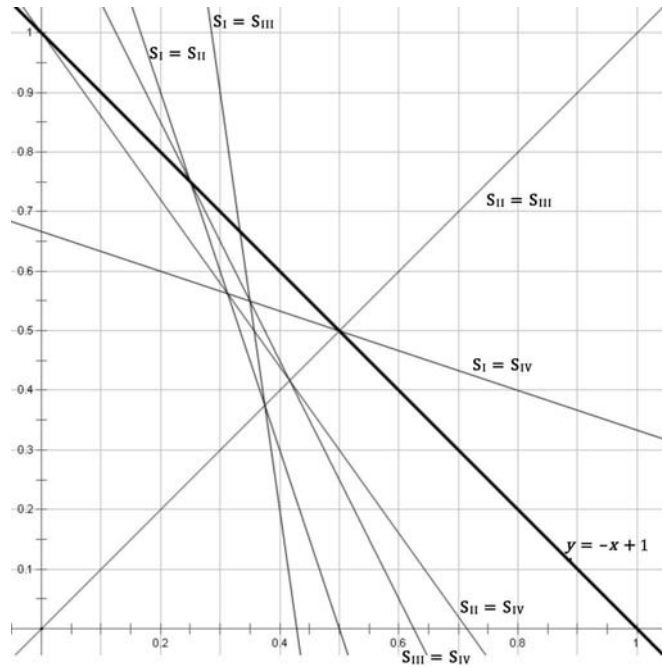
There is a good chance that this extension of the module could be of tactical value to a student. Suppose she really wants to go to College III, but has pressure from outside sources to choose a different one. Rather than choosing her preference outright, it might be more impressive and help her make her case to say, "Well, I set up my decision matrix, and then I made the choice that financial aid was 2/10 of my personal weight, academics was 5/10, and location was 3/10. When I set $x = 0.2$ and $y = 0.5$, I ended up smack in the middle of the region in which College III was the best! It just happened!"

When you look at the picture, however, you see that it would be much more difficult to end up with College IV as the best choice. There is only a small triangular region in which College IV is preferred to each of the other three, and you would have to pick something very near $x = 0.35$ and $y = 0.6$ (and therefore $z = 0.05$) to end up there.

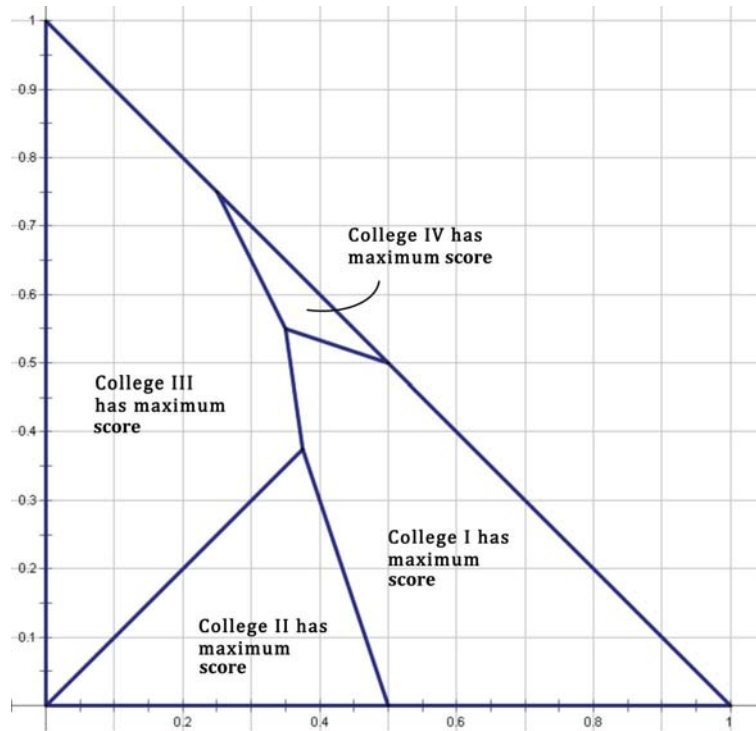
Note that it is possible to use a different coordinate system so that all three of x , y , and z can be seen at the same time. These are called *barycentric coordinates*, they are not well known, and it would take a major project to see how they work.

CHOOSING A COLLEGE

Teacher's Guide — Extending the Model



A graph of each of the combinations of scores set equal to one another so that $S_i = S_j$ and one of the lines defining triangle, T.



A graph showing each of the four polygonal regions in which each of the colleges can attain a maximum score compared to the others.

A TOUR OF JAFFA

Teacher's Guide — Getting Started

Inbar Aricha
Hadera, Israel

Purpose

In this two-day lesson, students will model a graph optimization problem called the “Traveling Salesman Problem” (TSP). The TSP seeks to minimize the cost of the route a salesperson should follow to visit a set of cities and return to home. The goal is to find a minimal-cost *Hamilton circuit* in a *complete graph* having an associated cost array, M .

To begin, explain the situation to students. They are about to visit a new place such as a zoo, a city, a shopping center, or an amusement park, and they wish to plan their trip beforehand. What should they consider when planning their trip? How would they plan the most efficient route?

Prerequisites

Students need only basic understanding of graphs and matrices or arrays.

Materials

Required: Rulers.

Suggested: Push-pins, corkboard, and string.

Optional: Internet access, printer (to find and print maps of different attractions).

Worksheet 1 Guide

The first three pages of the lesson constitute the first day's work. Students are asked to think of a site they wish to visit — or, in the absence of availability of a computer with internet access, they may use the map of Jaffa provided. Students identify 5–7 sites that are “must-sees” in that they are the most important to visit while on the trip. Students consider different variables that should be taken into account when planning a trip to that site; these variables include distance or time to travel from one site to another, or perhaps the cost to use a toll-road on the route between these areas. Students build their own model for the problem of planning the best route for their visit at their chosen site. They are then introduced to the model a mathematician would generally build, a **graph**. Finally, they are challenged to find a “best route” using the graph and must consider if this is, indeed, the best route possible.

Worksheet 2 Guide

The fourth and fifth pages of the lesson constitute the second day's work. Students are given the definition of a **Hamilton circuit** as well as an algorithm to find efficient Hamilton circuits in mathematical language. Students will be asked to think deeply about the properties and constraints of the model they created. Students then apply the algorithm and use a cost array to determine how close the algorithm came to the lower bound of the route.

CCSSM Addressed

N-Q.2: Define appropriate quantities for the purpose of descriptive modeling.

G-MG.3: Apply geometric methods to solve design problems.

A TOUR OF JAFFA

Student Name: _____ Date: _____

Have you ever been on vacation and didn't get to visit all the attractions you wanted? Do you think you could have used mathematics to help you get to all or, at least, more of the attractions you wanted to visit?



Map illustration by Comap

THE CLOCK TOWER SQUARE

- A- THE CLOCK TOWER. One of a hundred clock towers erected throughout the Ottoman Empire in 1900, commemorating the twenty five years of the Sultanate Abdul Hamid the Second. The Clock Tower was the focus point for the diverse commercial activities and many markets flourished around it.
- B- THE SARAYA. The Turkish Government building, in the center of the market square (known today as the Clock Tower Square), was erected in 1897. Saraya means castle in Turkish.
- C- THE MAHMUDIYA MOSQUE. Jaffa's large Mosque, built by Muhammad Abu Nabut, who was the Ottoman ruler of the city between 1807–1818. Nabut was responsible for building and developing Jaffa after a long period of recession. On the southern side of the mosque is the SABIL (meaning road)–water fountain, where travelers and their livestock stopped to refresh themselves before continuing their journey.

OLD JAFFA

- D- ST. PETER'S CHURCH. Built by the Franciscan Church between 1888-1894. As early as the 17th century, Franciscan Monks arrived at this site and built a church on the remains of a crusader fortress dated from the days of King Louis IX, who took part in the Crusades. According to local tradition, the church also hosted Napoleon when he visited Jaffa on his journey in the land of Israel.
- E- HOUSE OF SIMON THE TANNER. The site has real importance in the Christian tradition. In his house stayed St. Peter (one of the foremost apostles of the Christ and also considered the first pope) and there took place the miracle of the dream.
- F- THE BRIDGE OF DREAMS. According to an ancient legend, wishes will be granted to anyone who stands on the bridge, holding his astrological sign and looking at the sea.

Leading Question

How can you plan a route so you have time to make it to all the sites you want to see?

A TOUR OF JAFFA

Student Name: _____ Date: _____

Choose a specific site that you wish to visit, such as a zoo, a theme park, a shopping mall, a recreation park, a new city, or any other site you can think of. Make sure that the site you choose has several points of interest.

1. You will not be able to visit *all* the attractions at your site, so choose 5–7 of your favorite points of interest. How many ways are there to travel from any starting point you choose, visit each site exactly once, then return to the starting point?

Why do you think you should only visit each place (except the starting/ending point) exactly once? Is it necessary to start and end at the same place?

2. What do you think is a good way to plan your route? What might cause you to be unable to visit all the sites you want in a single day? What do you need to know about the site before you can plan a route?

3. Make a mathematical model to help you plan your route.

What do you think is important to consider in your model? What do you think you can “ignore” for now?

A TOUR OF JAFFA

Student Name: _____ Date: _____

4. Did your model help you find a route? Did it help you find *the best* route? How can you be sure? Is there a way to be sure? Explain.

One way that mathematicians would show a route is to use a **graph**. These are not the types of graphs that you usually think of, though. These graphs have two important features: **vertices** (these usually are drawn as points or dots and they represent something of interest; the singular form of the name is **vertex**) and **edges** (lines connecting the vertices; they are used to show some relationship between the vertices they connect).

5. Did you use a graph to model your route or not? If not, try to do so. What do the vertices represent? What do the edges represent? Which model do you like better and why? If you did make a graph, explain how you chose your vertices and edges. What do they represent?

6. Are there factors that you ignored while making your graph that maybe you shouldn't have? Is there a way to modify the graph so some of these factors can be considered? If so, modify it.

Are all the edges equal or do they have different "edge weights"? What might "edge weight" mean?

7. Are you sure that you found the best, most *efficient* route? What does it mean for a route to be the "most efficient"? Give an example of two different routes from your graph. Is one more efficient than the other? How can you tell?

A TOUR OF JAFFA

Student Name: _____ Date: _____

A route (known as a **path**) that starts and ends at the same vertex and visits each vertex in the graph exactly once until ending at the starting point is called a **Hamilton circuit**, and the problem of looking for the most efficient Hamilton circuit is a famous mathematical problem called the *Traveling Salesman Problem (TSP)*. There is an algorithm (a set of steps) to help find some very efficient Hamilton circuits.

The algorithm has three requirements:

- i) The cost of going between two vertices is the same in either direction. (The cost is **symmetric**.)
- ii) The cost of going from vertex A to vertex B is less than or equal to the cost of going from vertex A to vertex C to vertex B. (The costs fulfill the **triangle inequality**.)
- iii) Each vertex is connected by an edge to every other vertex. (The graph is **complete**.)

8. Explain why and in which real-life cases these requirements are reasonable.

9. Does your graph fulfill these requirements? How would you make it fulfill the requirements without drastically changing what you expect to be the most efficient paths?

A TOUR OF JAFFA

Student Name: _____ Date: _____

The Traveling Salesman Algorithm

The following algorithm helps to determine **near-minimal** routes.

- I) Pick any vertex as a starting point for a circuit C_1 consisting of 1 vertex.
- II) Given the circuit C_k with k vertices and $k \geq 1$, find the vertex Z_k not in C_k that is closest to a vertex in C_k ; call the vertex in C_k that is Z_k is closest to Y_k .
- III) Let C_{k+1} be the circuit with $k+1$ vertices obtained by inserting Z_k immediately before Y_k in C_k .
- IV) Repeat steps II and III until a Hamilton circuit (containing all vertices) is formed.

10. Can you use an array M to represent the cost of moving between any two vertices on your graph? What does the entry in the cell $M(i,j)$ represent? What is the value of a cell $M(i,i)$?

11. Apply the algorithm for several different starting points. Compute the “cost” of the route you found. Did you get the same cost for each route?

12. Look at your graph and cost array and try to find a lower bound for the optimal route. How close is the lower bound to the smallest result from the algorithm? Do you think the algorithm got you reasonably close to the lower bound? Is the algorithm a good way to help you visit all the places you want to visit?

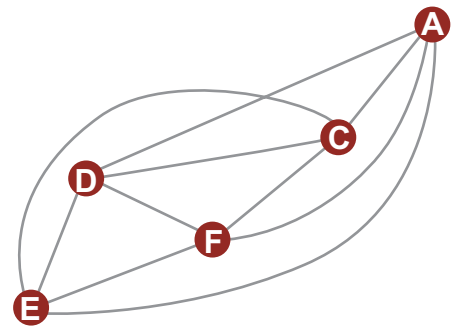
A TOUR OF JAFFA

Teacher's Guide — Possible Solutions

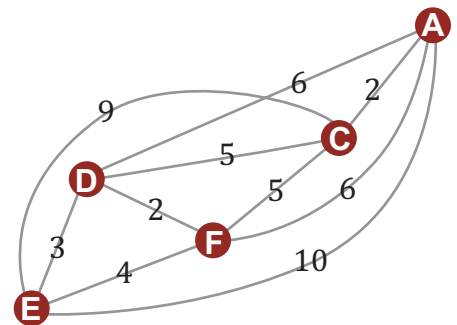
The solutions shown represent only some possible solution methods. Please evaluate students' solution methods on the basis of mathematical validity.

1. A tour of old Jaffa in Israel could feature the following five sites: A – The Clock Tower, C – The Mahmudia Mosque, D – Saint Peter's Church, E – House of Simon the Tanner, and F – The Bridge of Dreams. There are five different attractions and so there are $5! = 120$ different routes. For n attractions, there are $n!$ different routes. It is usually important to start and end at the same point, since we'll usually park our car and will want to start and end next to it. You can also plan a trip in which, for example, you use public transportation and can start and end at different points.
2. There's a good chance that there won't be enough time to visit all the attractions or there may be monetary limitations due to entrance fees and problems of the like. Some factors to check between any two attractions are distance, time, money, different modes of transportation (such as walking, driving a car, or public transportation), and the cost of each attraction.

3. The undirected graph to the right can be used to model these attractions.
4. The model helps to see the different possible routes, but it's difficult to decide which is "best".
5. The vertices represent the attractions. The edges represent routes between vertices.
6. Edges are not equal; each edge can represent any of the factors suggested in the answer to question 2. "Edge weight" is a number that is assigned to each edge that represents the factor chosen. On the graph shown to the right, each edge weight represents the time it takes (in minutes) to go from one site to another. Another choice for edge weights could have been distance between two sites.



7. The most efficient route is the route that goes through all vertices at a minimal time. This is because time is the important factor chosen. If distance was highlighted as the edge weight meaning in question 6, then the most efficient route would have been the shortest one that goes through all the vertices once and only once. Essentially, out of all possible routes, the sum of the edges that are used in the most efficient route will be the smallest. In the graph shown, the route AED-FCA takes 22 minutes. The route CAFEDC takes 20 minutes. The route CAFEDC is more efficient. It is unclear if the latter is the most efficient route. Without an algorithm to insure the most efficient route, one would have to check all 120 possible routes.



8. (i) A graph that models this problem can be directed (with non-symmetric costs) or undirected (with symmetric costs). One real-life case is when edge weights represent distance, walking can be represented as an undirected graph, while it will be more reasonable to represent driving with a directed graph; when the edge weights represent time, walking in a level plane can be represented as an undirected graph, while it will be more reasonable to represent driving and or walking on an unlevel plane by a directed graph.

A TOUR OF JAFFA

Teacher's Guide — Possible Solutions

(ii) When the edge weights represent distance or time the triangle inequality is satisfied.

(iii) If the graph is not complete, adding an arbitrarily long edge when there is no path between two attractions will complete the graph without affecting the optimal route.

9. The graph in question 6 fulfills constraints i and iii, but not ii; $AE = 10$, but $AD + DE = 9$. The probable source of this error is that time was rounded to whole minutes and the sum of two numbers that round to 6 and 3 respectively can easily round to 10. This can be solved by changing the weight of edge AD to 9 and then the graph will fulfill the second constraint.

	A	C	D	E	F
A	∞	2	6	10	6
C	2	∞	5	9	5
D	6	5	∞	3	2
E	10	9	3	∞	4
F	6	5	2	4	∞

10. The associated cost array is shown to the right. The value of $M(i,j)$ is the cost of using the edge from vertex i to vertex j . The value of $M(i,i)$ is the cost of using the edge from the vertex i to itself, and so, doesn't matter; one can use the infinity symbol or any other symbol or notation that won't be confused as a possible value between two vertices.
11. Start with vertex A as C_1 . Vertex C is closest to A, so $C_2 = ACA$. Vertices D and F are the vertices not in C_2 that are closest to vertices in C_2 , namely, closest to C. Pick F, so $C_3 = AFCA$. Now, vertex D is the vertex not in C_3 that is closest to a vertex in C_3 , namely, closest to F; thus $C_4 = ADFCA$. Finally, vertex E is of distance 3 from D, so place it before D. A near-minimal route has been obtained: $C_5 = AEDFCA$, whose cost is 22. Try other vertices as the starting vertex or other decisions (in cases of more than one option) and apply the algorithm again; other near-minimal routes are obtained. Pick the shortest of these. So, starting with C, the near-minimal route CAFEDC, whose cost is 20, is obtained; starting with D, the near-minimal route DFEACD, whose cost is 23, is obtained; starting E, the near-minimal routes EACFDE and EFACDE, whose costs are 22 and 23 respectively, are obtained. The best result so far costs 20 minutes.

A TOUR OF JAFFA

Teacher's Guide — Possible Solutions

12. A lower bound for the cost of this TSP can be obtained by subtracting a constant value (as large as possible) from every row and then from every column without making any entry in a row or a column negative. This works because every route must contain an entry in every row/column, the edges of a minimal tour will not change if we subtract a constant value from each row/column of the array. In this case, subtract a total of $2+2+2+3+2+1=12$ to obtain the array shown. A minimal route using the cost in this array must cost at least 0, and so a minimal route using the original array will cost at least 12. In general, the lower bound for the TSP equals the sum of the constants subtracted from the rows and columns of the original cost array to obtain a new cost array with a 0 entry in each row and column.

	A	C	D	E	F
A	∞	0	4	7	4
C	0	∞	3	6	3
D	4	3	∞	0	0
E	7	6	0	∞	1
F	4	3	0	1	∞

A TOUR OF JAFFA

Teacher's Guide — Extending the Model

First of all, what if students wish to go beyond question 12, which gives a lower bound of 12? Some might find it interesting to consider an argument like the following: it sure looks like using the link from A to C is a good idea since it gives no additional cost than the minimum necessary (that's what the 0 in the array means). If you use AC, your circuit will have to *get into* A from somewhere, and the cheapest link into A is from D or F at a cost of 4. Similarly, the circuit will have to *go out of* C to somewhere (but not A), and the cheapest way out of C is to either D or F, which cost 3. So in fact, you know that a circuit which uses AC will cost at least $12+4+3$, which is 19. As a matter of fact, 19 is a darn good guess for the best possible answer.

Because the problem is so small — it has only 5 sites — there are only 12 possible circuits, namely ACDEFA, ACDFEA, ACEDFA, ACEFDA, ACFDEA, ACFEDA, ADCEFA, ADCFEA, AECDDFA, AECDFDA, AFCDEA, and AFCEDA. (Twelve others are each of these read in reverse.) You can compute the costs of these 12 circuits from the table in problem 10. See how close you get to 19.

Wait a minute! Question 1 said that there are $n!$ different circuits, and with $n = 5$, this is 120. How come only 12? Well, when you have n sites, the $n!$ comes from starting at any site, then going to any other, etc., until you've been to them all, and then going back to the original site. But you will get the identical circuit n times by starting at any of the n sites. So there are only $(n-1)!$ directed circuits. And also we are assuming that our cost matrix is symmetric, so each circuit at each beginning can be traversed backwards at the same cost. That's where you obtain $(n-1)!/2$ for the number of undirected circuits. So really, the number of different circuits depends on how you're defining "different". By the way, just how do you make that list of twelve? How do you know it's right? Well, one way is to start and end with A and make sure that C is either second or third. Those lists in which C is fourth or fifth are then the ones given when you read each of them in reverse order.

The instinct for what goes on in a TSP often comes from a TSP in the Euclidean plane. In that case, the costs are Euclidean distances and according to question 7, the distances are symmetric, there is a known distance between any two sites, and the distances satisfy the triangle inequality. That, as we have said, is where our instincts come from. It follows that the circuit never passes through any site more than once, and the circuit never crosses itself. (That's a theorem.)

Contrary to instinct, the cost in the real world is not necessarily direct Euclidean distance; it may be something like distance along actual streets or pathways, or may be time along the pathway rather than distance, for example. This is what happens in the Jaffa problem. You can't go "as the crow flies" from one point to another; there may be walls, buildings, ditches, and other obstacles in the way. You also may have noticed that the costs in the table of question 10 do not all satisfy the triangle inequality. AD costs 6, DE costs 3, but AE costs 10. What's going on here? Well, the physical route for walking from A to E probably goes through D. And let's imagine that the costs were time, and the original numbers perhaps had one more significant figure, so that AD was really 6.3 minutes, which rounds to 6 minutes; DE was 3.4 minutes which rounds to 3 minutes, and AE was 9.7 minutes, which rounds to 10 minutes. Maybe that's why it looks like the triangle inequality was violated. All are perfectly real, but that's the kind of difficulty you have to watch out for! And the circuit ACDEFA is a very good one, but if you look at it as the crow flies on the tour map, it crosses itself! But you couldn't walk exactly that way! Euclidean distance is a good guide, but the numbers in the real world are not exactly proportional to Euclidean distance.

If you are thinking of a TSP for an airplane business trip, the situation with airplane fares is much worse! Between many pairs of commercial airports there are no direct flights and you will pay for the actual routing — or worse. Flying to, or through, airports in which there is lots of competition is usually cheaper than flying to a single-provider airport — and Euclid doesn't have anything to say about that!

GAUGING RAINFALL

Teacher's Guide — Getting Started

Stuart Weinberg
Teachers College, Columbia University

Purpose

In this two-day lesson, students will estimate the average rainfall for a 16 km by 18 km territory in Rajasthan, India. Rainfall estimations will be based on rain gauges scattered around the territory. Since these placements are varied, students will need to identify each gauge's "region of influence" to estimate the average rainfall.

To begin, explain the situation that needs to be modeled. Meteorologists need to understand average rainfall totals in a region in order to make short-term forecasts. These are usually for relatively shorter periods of time. Climatologists need to understand average rainfall totals for relatively longer periods of time in order to understand, among other things, climate change.

Prerequisites

Students need to understand equidistance, how to compute areas of various polygons, and how to compute a weighted average. The ability to make basic straightedge and compass constructions is desirable.

Materials

Required: Rulers.

Suggested: Compasses or protractors and colored pencils (to distinguish different polygons).

Optional: Geometry software or Internet access.

Worksheet 1 Guide

The first three pages of the lesson constitute the first day's work. Students are given the opportunity to explore their intuition regarding rainfall and suggest ways to approximate average rainfall. Students are likely to use an outright arithmetic mean to determine average weekly rainfall. Maps are then introduced to convey the idea that the relative placement of each of the gauges is mathematically important. Finally, students are asked to try to construct a model that will take into account the placement of the gauges.

Worksheet 2 Guide

The fourth and fifth pages of the lesson constitute the second day's work. To begin, students consider the idea of "region of influence". Voronoi diagrams (also called Thiessen polygons in relation to meteorology) are introduced. Students are asked how Voronoi diagrams may be useful in estimating average depth of rainfall and will construct these diagrams. They are then asked to determine how the polygons can be used to weight the readings at the rain gauges and will use this method. Then the model is extended to use more gauges. The students will determine which of their original method (from the first day) and Voronoi diagrams works better than the other or if they work in the same way. Finally, students are asked to consider the main property of the polygons in Voronoi diagrams (the boundaries of regions of influence) and determine where else they can be applied. Students may want to research possible uses on the internet.

CCSSM Addressed

N-Q.3: Choose a level of accuracy appropriate to limitations on measurement when reporting quantities.

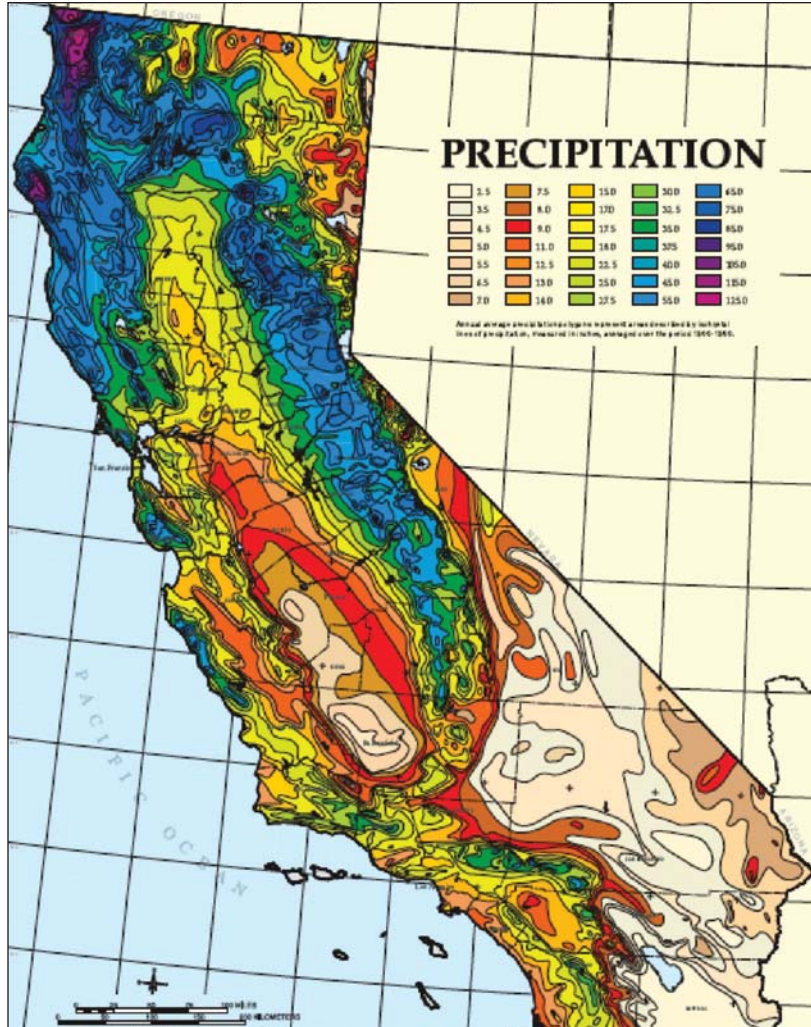
G-MG.1: Use geometric shapes, their measures, and their properties to describe objects.

G-MG.3: Apply geometric methods to solve design problems.

GAUGING RAINFALL

Student Name: _____ Date: _____

Meteorologists and climatologists are concerned with tracking the amount of rainfall in a given place over different periods of time. They use these data for things like making short-term forecasts and making long-term inferences about climate change. They collect rainfall data using rain gauges that are spread out around the region that they are studying.



Source: California Precipitation Map — Department of Forestry and Fire Protection

Leading Question

How can a climatologist determine the average rainfall using rain gauges spread throughout a territory in the state of Rajasthan in India? The territory is rectangular, measuring about 16 km by 18 km and the gauges are scattered around.

GAUGING RAINFALL

Student Name: _____ Date: _____

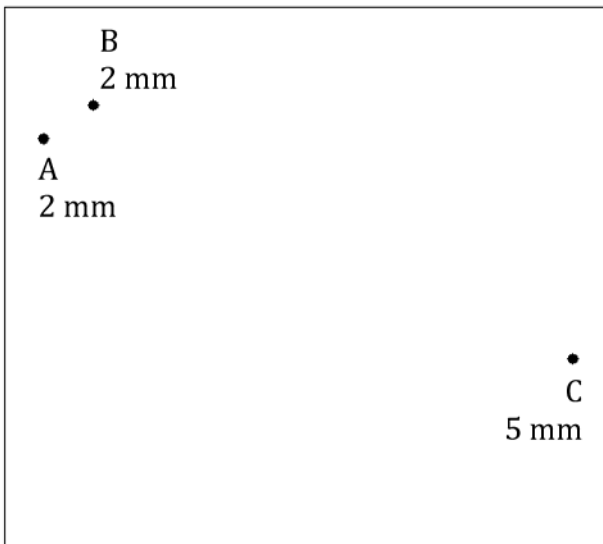
1. Consider the territory described in the leading question. Use the table below to determine the average rainfall for the territory in that week. The table below gives the rainfall measurements for one week at each gauge.

Gauge	Rainfall Depth (in mm)
A	12.6
B	13.4
C	10.8

2. Using the data from question 1, determine the total volume of rainfall for the week. Can this be done? Why or why not? Explain.

Do you need more information? What is meant by "total volume"? What do you know about volume?

3. Suppose in another week, the rain gauges give the total rainfall as in the map below. What is the average depth of the rainfall that week?

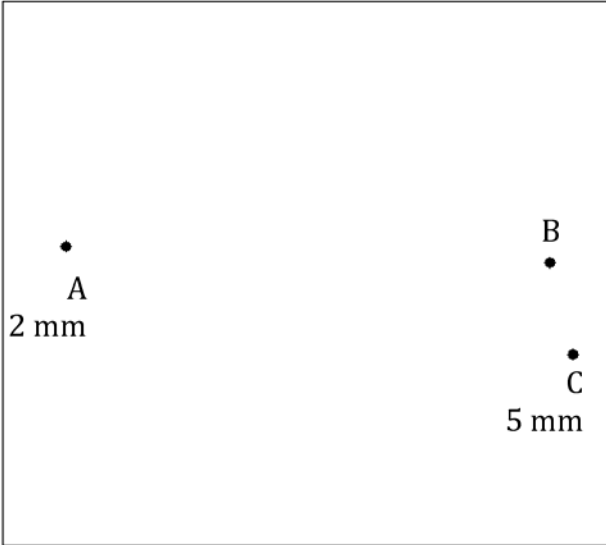


GAUGING RAINFALL

Student Name: _____ Date: _____

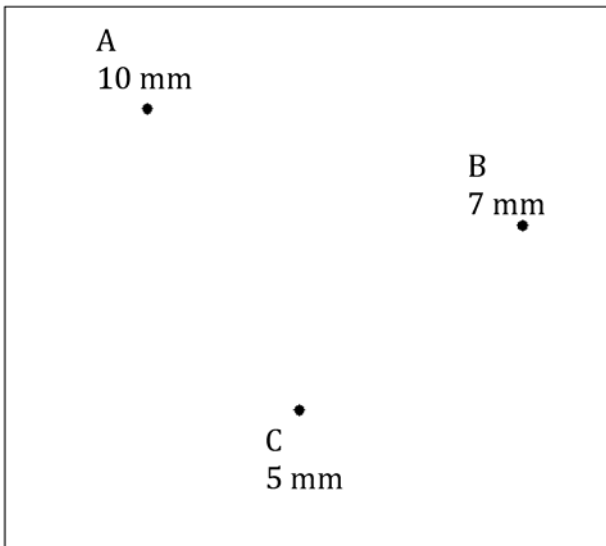
4. Consider the rainfall in another week in this territory. How much rainfall do you think was measured at the gauge at B? Why do you think that? Using your guess for the depth of rainfall at B, find the average rainfall in the territory.

For what points is the gauge at A a better estimator? At C? Is B in one of those regions?



5. What do you think your answers for questions 3 and 4 say about rainfall gauges? What is important to consider when looking at the measurements from the gauges?

6. Consider the map of the territory below. Use your ideas from question 5 to help you create a better model to estimate the week's average depth of rainfall in the territory below.



GAUGING RAINFALL

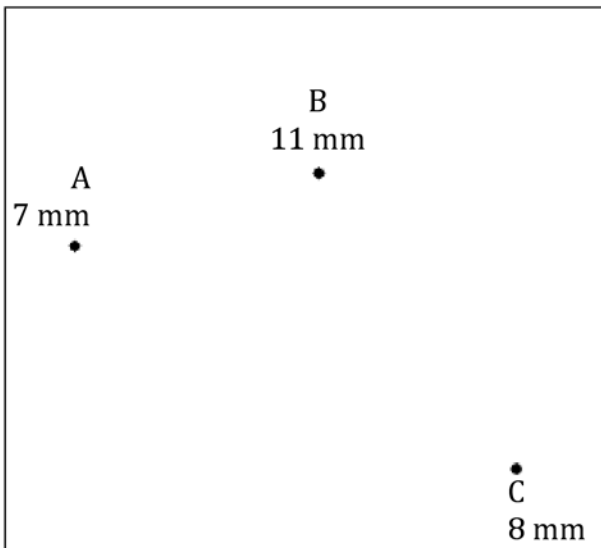
Student Name: _____ Date: _____

7. It seems that rain gauges have different “regions of influence” depending on where they are placed. Did your model from question 6 use “regions of influence”? How might using these help you estimate the average weekly rainfall?

In mathematics, a **Voronoi diagram** is a partition of a space as a set of discrete polygons. Each region contains one “center of influence”. The other points in the interior of a polygon represent all the points that are closer to that polygon’s point of interest than any other point of interest. In meteorology, Voronoi diagrams are also called Thiessen polygons.

8. Why does a Voronoi diagram help to determine the average depth of rainfall?

9. Use Voronoi diagrams to estimate the average weekly rainfall in the map given below.

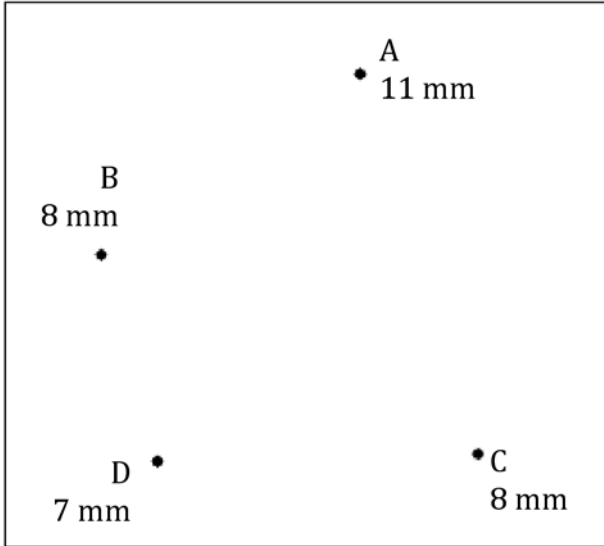


How would you construct a Voronoi diagram? What is important about their boundaries?

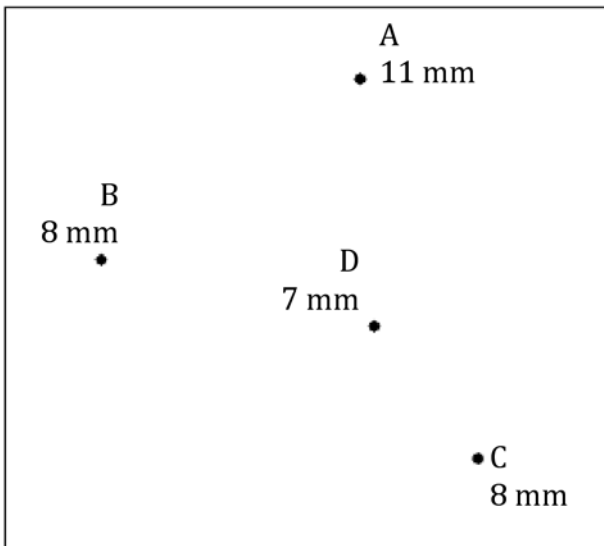
GAUGING RAINFALL

Student Name: _____ Date: _____

10. Use two methods to estimate the average weekly depth of rainfall for the map below: first, your method from question 6, and second, Voronoi diagrams. Did both methods give the same result? Which method seems to work better?



11. Use the method you used in question 6 and also Voronoi diagrams to estimate the average weekly depth of rainfall for the map below. Do both methods work here?



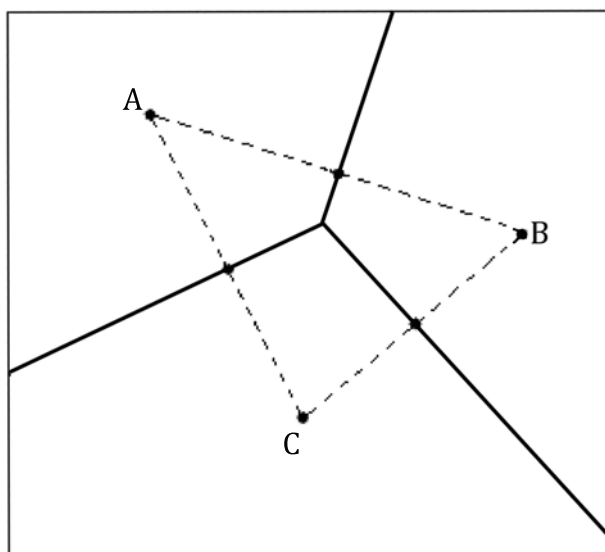
12. Where else do you think you can use Voronoi diagrams? What property of Voronoi diagrams makes them reasonable to use in these types of applications?

GAUGING RAINFALL

Teacher's Guide — Possible Solutions

The solutions shown represent only some possible solution methods. Please evaluate students' solution methods on the basis of mathematical validity.

1. Arithmetic Mean: $(12.6 \text{ mm} + 13.4 \text{ mm} + 10.8 \text{ mm})/3 = 12.27 \text{ mm}$
2. Area of region = $(16 \cdot 18) \text{ km}^2 = 288 \text{ km}^2$. Volume $\approx 3,500,000 \text{ m}^3$. (Change units.)
3. Arithmetic Mean: $(2 \text{ mm} + 2 \text{ mm} + 5 \text{ mm})/3 = 3 \text{ mm}$.
4. B probably measured 5 mm. This is because rainfall does not change much over short distances. Average rainfall will thus be $(2 \text{ mm} + 5 \text{ mm} + 5 \text{ mm})/3 = 4 \text{ mm}$.
5. Rainfall gauges near each other will have similar rainfall totals. The relative position of each gauge is an important variable to consider.
6. The Voronoi diagram (to the right) will give the polygons that represent each gauge's "region of influence". They are constructed using the perpendicular bisectors of each side of the triangle ABC. Average rainfall is computed by using the relative area of each polygon and multiplying this by the rainfall at the gauge encompassed by the polygon, then summing. The areas for the polygons defined by A, B, and C represent 0.301, 0.313, and 0.386 of the total area, respectively. So the average rainfall is about 7.1 mm.
7. "Regions of influence" can be used in a weighted average of the rainfall.
8. The Voronoi diagram helps determine average rainfall because polygon boundaries represent all points equidistant from two points and their interiors represent all points closest to its gauge than any other gauge.
9. Using the same model as in question 6, the average depth of rainfall is about 8.4 mm.
10. Answers will vary, but the average depth of rainfall for the Voronoi diagram method is about 8.8 mm. The proportion of areas given by A, B, C, and D are about 0.314, 0.244, 0.257, and 0.185, respectively.
11. Answers will vary, but the average depth of rainfall should be about 8.5 mm. The Voronoi method should work as a good approximation; the specific model the student initially chose to use may or may not work as well.
12. Voronoi diagrams can be used to calculate the end of a solar system because the end of a solar system is the boundary at which the star's influence is less than the next closest star's influence. They have also been used in anthropology to determine the influence of Mayan city-states and by epidemiologists to show a point of origination of disease.

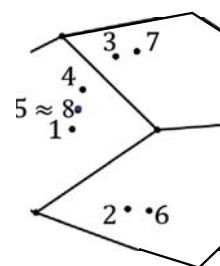


GAUGING RAINFALL

Teacher's Guide — Extending the Model

The ideas in this model have been used in building understanding in a surprisingly large number of situations. One of the earliest that is often cited was in determining a likely source of the Broad Street cholera outbreak in London in the mid-1850s. It was determined that each of a large number of victims lived closer to a particular water pump than to any other and this pump was then determined to be the source of the infection. But there are dozens of other applications — in chemistry, in archaeology, you name it.

For those interested in going further into the geometry of Voronoi diagrams, Chapter 5 of Course 2 in COMAP's *Mathematics: Modeling Our World* contains a number of suggestions. An especially nice problem is to recover the centers of each individual region given the boundaries, as distinguished from the original problem of finding the polygon boundaries given the centers. One method that might particularly appeal to those with an interest in algorithms is as follows: Pick a location X that seems likely to be close to the center you are trying to find. Then its reflection in one of the edges should be close to the center of that polygon. Keep doing this as you "go around" a corner at which polygons meet until you get to your original polygon. If $X' = X$, then your direction from the corner A is correct. If $X' \neq X$, pick a new guess halfway in between X' and X and try again. You will rapidly approach the correct direction from A . Then do the same process around an adjacent corner B . You then know the direction from A to the center and the direction from B to the center. Together they determine the location of the center. (See the image shown to the right with the points numbered from 1 to 8 in the order they were placed and/or reflected around corner A .)



We are now within sniffing distance of a computer algorithm, and for those who are doing a first course in computer science, here is one more connection to Voronoi diagrams. Your course is likely to include two methods for finding a shortest connecting network, also called a **minimal spanning tree** in the context of graph theory. The two methods are Kruskal's Method and Prim's Method, and let's look at them for vertices in the Euclidean plane. In both methods, it is necessary at some time to compute the distance from every vertex to every other. (In Kruskal's, you do them all at once; in Prim's, you do them in dribs and drabs, but you do them all eventually.) Now, if the computer had eyes, it would know that two vertices which are far apart, with other vertices in between, never end up being directly connected. *You* know that, but how does *the computer* know that? A lovely result is the following: before you try to compute the shortest connecting network for your vertices, first compute the Voronoi regions for these vertices. Then vertex a can be connected to vertex b in a shortest connecting network *only* if the Voronoi polygons centered at a and at b share an edge! This is how the computer can "see" that two vertices are too far apart to be directly connected in a shortest connecting network.

As a practical matter, computing the Voronoi diagrams for a given set of vertices is not cheap, but for large numbers of vertices it is quicker than the full Kruskal and Prim algorithms. So it would pay you to do this computation – but only for a problem with many vertices.

Reference

Mathematics: Modeling Our World, Course 2 (2nd ed.). (2011). Bedford, MA: COMAP.

NARROW CORRIDOR

Lay Chin Tan
Singapore

Teacher's Guide — Getting Started

Purpose

In this two-day lesson, students are asked to determine whether large, long, and bulky objects fit around the corner of a narrow corridor.

The objective of this lesson is to apply the concept of turning points (maximum or minimum points) and the Pythagorean Theorem to determine the longest object that can go around the corner of a corridor.

Prerequisites

Students should know how to draw and interpret graphs and should know how to identify the maximum and minimum points of a graph. Prior knowledge of Pythagorean Theorem is required.

Materials

Required: Ruler (metric).

Suggested: A graphing calculator or other graphing utility.

Optional: None.

Worksheet 1 Guide

The first four pages of the lesson constitute the first day's work in which students are introduced to the problem of moving a sofa, but then asked to investigate a similar but simpler problem. Instead of a sofa, which is a three-dimensional object, they are asked to explore the case where a plumber tries to carry a long pipe around a corner. Since no prior knowledge in differentiation is necessary, students are expected to use graphing tools to sketch the graph of the mathematical expression that they have formulated. They are then required to interpret the graph(s) and draw a conclusion. This activity can be modified to incorporate differentiation to find the minimum value of a function.

Worksheet 2 Guide

The fifth and sixth pages of the lesson constitute the second day's work in which students use the results obtained in the earlier class to model the original problem. Different corridor shapes are introduced to incorporate real-world variations within their model.

CCSSM Addressed

A-CED.1: Create equations and inequalities in one variable and use them to solve problems.

A-CED.2: Create equations in two or more variables to represent relationships between quantities; graph equations on coordinates axes with labels and scales.

F-IF.7: Graph functions expressed symbolically and show key features of the graph, by hand in simple cases and using technology for more complicated cases.

F-BF.1: Write a function that describes a relationship between two quantities.

G-SRT.8: Use trigonometric ratios and the Pythagorean Theorem to solve right triangles in applied problems.

NARROW CORRIDOR

Student Name: _____ Date: _____

George and Linda wanted to buy a sofa for their new apartment at a sale. Linda saw a sofa that she really liked. But George thought otherwise.



The sofa is 3 feet wide, 9.5 feet long, and 3 feet high. Figure 1 shows the floor plan of the corridor that leads to George and Linda's new apartment. In addition, the ceiling is 9 feet above the floor.

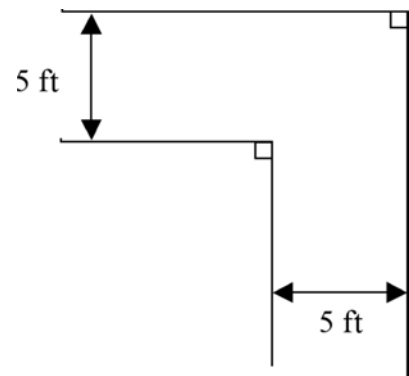


Figure 1

Leading Question

If George and Linda buy the sofa, will they be able to move it into their apartment?

NARROW CORRIDOR

Student Name: _____ Date: _____

Before modeling with a sofa, think of a similar problem in which a plumber tries to carry a long pipe horizontally around the corner of the corridor. You may assume that the width of the pipe is negligible. If the pipe is too long, it will be stuck at the corner as shown in Figure 2.

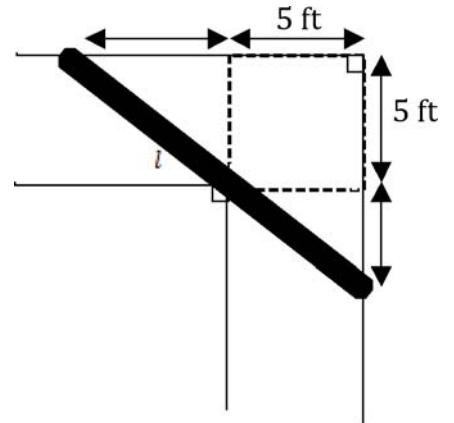


Figure 2

- Investigate the relationship between l , y , and x in Figure 3 on the next page. Complete the following table by measuring l and y with a ruler for different values of x .

$a =$ _____ cm.

x cm	y cm	l cm
10		
9		
8		
7		
6		
5		
4		
3		
2		
1		

NARROW CORRIDOR

Student Name: _____ Date: _____

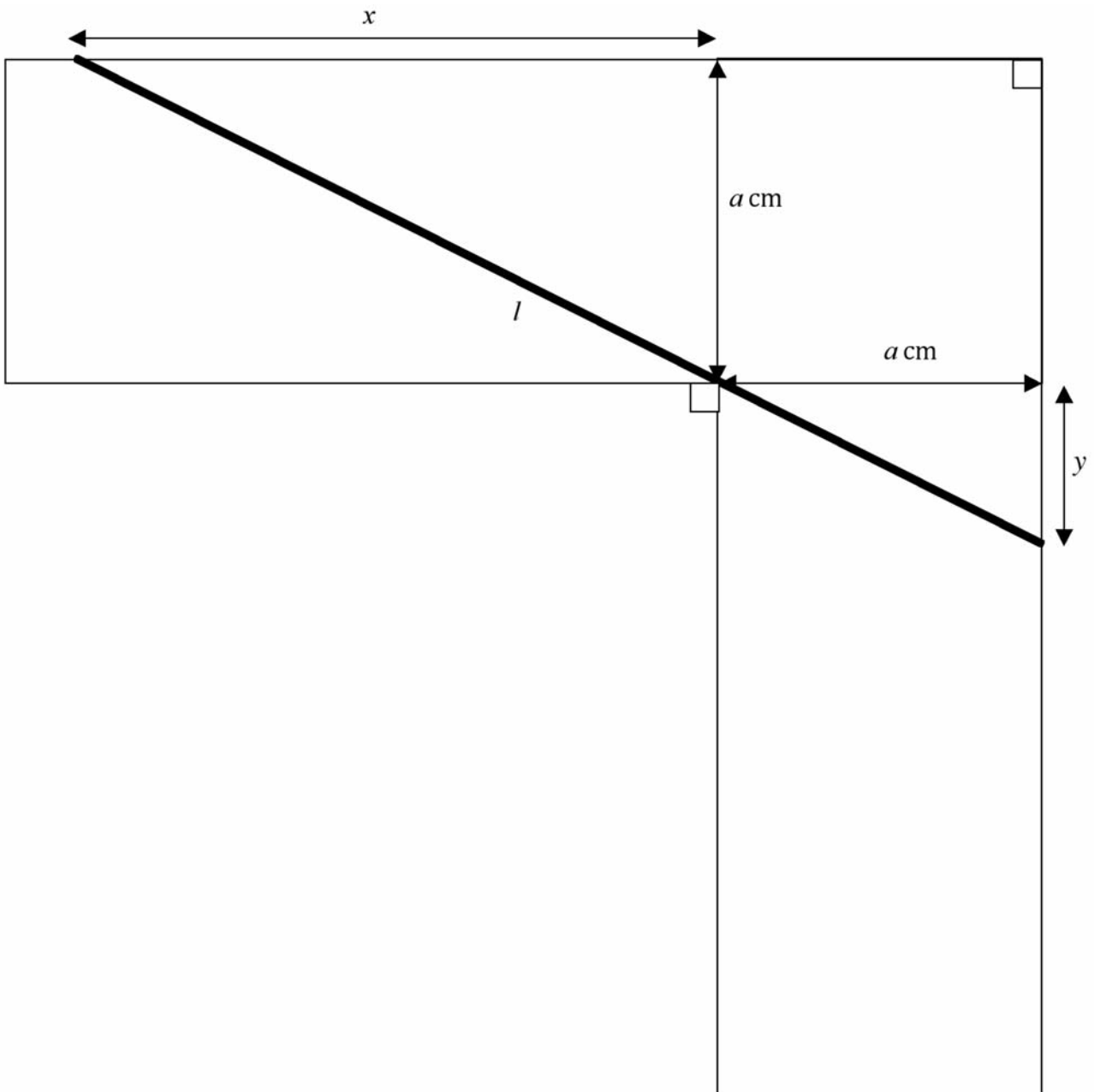


Figure 3

2. Use a graphing calculator to draw the scatter plot of l against x . What do you observe?

NARROW CORRIDOR

Student Name: _____ Date: _____

3. Write an algebraic expression for l in terms of x .
4. With the help of a graphing calculator, sketch the graph of the equation found in question 3 for $1 \leq x \leq 10$. What do you observe? Does the graph fit the scatter plot in question 2?
5. From the graph found in question 4, what is the length of the longest pipe that can go around the corner of the corridor horizontally in Figure 2?
6. If it is not necessary for the plumber to carry the pipe *horizontally*, do you still think the answer obtained in question 5 is the length of the longest pipe that can go around the corner? Justify your answer.

NARROW CORRIDOR

Student Name: _____ Date: _____

Use your previous results to help solve the original question of moving a sofa around the corner of a corridor.

7. What is the length of the longest sofa with a width of 3 feet that can go around the corner of the corridor horizontally?

8. If the movers are allowed to tilt the sofa while moving it, what is the length of the longest sofa that can go around the corner of the corridor? Do you think George and Linda should buy the sofa?

NARROW CORRIDOR

Student Name: _____ Date: _____

9. If the corner of the corridor makes an angle of 120° instead of a right angle as shown in Figure 4, what is the length of the longest sofa with a width of 3 feet and a height of 3 feet that can go around the corner? Should George and Linda buy the sofa in this case?

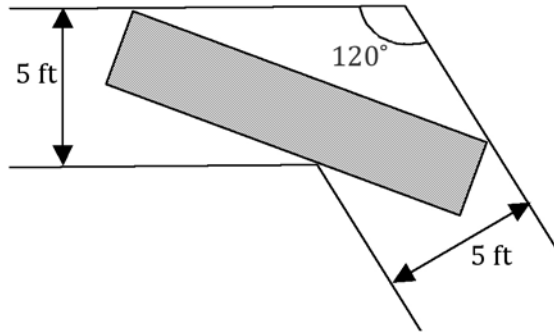


Figure 4

10. Suppose George's and Linda's apartment is along the corridor as shown in Figure 5 and the width of the door is 4 feet and its height is 8 feet. Will the longest possible sofa found in questions 7 and 8 be able to fit through the door?

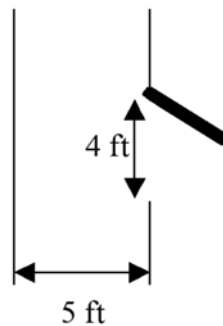


Figure 5

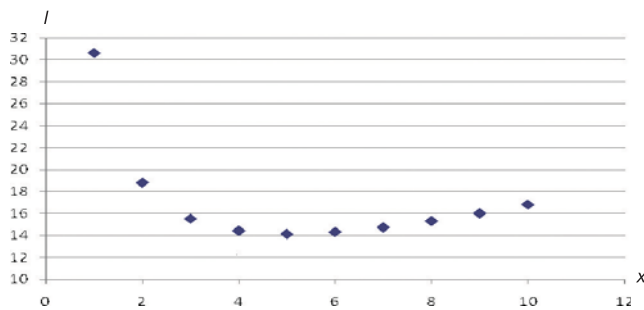
NARROW CORRIDOR

Teacher's Guide — Possible Solutions

The solutions shown represent only some possible solution methods. Please evaluate students' solution methods on the basis of mathematical validity.

- Students should obtain a value of approximately $a = 5$ cm.
- A scatter plot on their graphing calculator should look similar to the one pictured. They should conclude that there exists a minimum value of l as x varies. In other words, there exists the shortest pipe that will be stuck at the corner of the

x cm	y cm	l cm
10	2.5	16.8
9	2.8	16.0
8	3.1	15.3
7	3.6	14.7
6	4.4	14.3
5	5.0	14.1
4	6.3	14.4
3	8.3	15.5
2	12.5	18.8
1	25	30.6



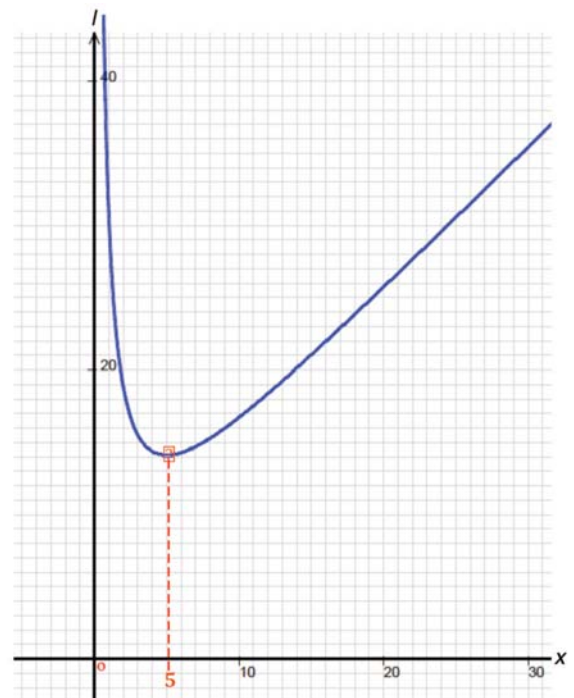
corridor and it appears to occur when $x = 5$.

- By the Pythagorean Theorem, $l^2 = (x + 5)^2 + (y + 5)^2$. By similar triangles, $\frac{y}{5} = \frac{5}{x} \rightarrow y = \frac{25}{x}$.

Therefore, $l^2 = (x + 5)^2 + \left(\frac{25}{x} + 5\right)^2$ which yields $l = \sqrt{(x+5)^2 + \left(\frac{25}{x} + 5\right)^2}$

- The graph of l against x for $x > 0$ has a minimum point at $x = 5$.

When $x = 5$, $l = \sqrt{10^2 + (5+5)^2} = 10\sqrt{2}$. When the graph is superimposed on the scatter plot in question 2, the graph should fit the scatter plot well. The solution can also be obtained by using trigonometric functions.



NARROW CORRIDOR

Teacher's Guide — Possible Solutions

5. From question 4, the length of the longest pipe that can go around the corner is $10\sqrt{2}$ ft.
6. Using the Pythagorean Theorem, the length of the longest pipe that can go around the corner is 16.8 ft.
7. The length of the longest sofa that can go around the corner horizontally is 8.14 ft. So the sofa, which is 8.5 ft. long, will not be able to go around the corridor horizontally.
8. The length of the longest sofa that can go around the corner of the corridor is 9.18 ft. George and Linda should not buy the sofa.
9. The length of the longest sofa that can go around the corner of the corridor is 10.2 ft.
10. If the sofa is moved horizontally, the length of the longest sofa that can pass through the door is 6.67 ft. The length of the longest sofa that can go around the corner of the corridor is 7.60 ft and so the sofa found in questions 7 and 8 will not be able to pass through the door.

NARROW CORRIDOR

Teacher's Guide — Extending the Model

The length of the longest pipe that would go around the 90° corner was computed by using the Pythagorean Theorem. The horizontal distance was $x + 5$, and the vertical distance was $(25/x) + 5$. Hence:

$$l = \sqrt{(x-5)^2 + \left(\frac{25}{x}-5\right)^2}.$$

The length l could have been computed in a different way. The pipe can be thought of as consisting of two pieces, one to the left of the point where it is up against the interior wall, and one to the right of that point. The piece to the left has length

$$\sqrt{x^2 + 25}$$

while the piece to the right has length

$$\sqrt{\left(\frac{25}{x}\right)^2 + 25}.$$

Thus, the length can also be written as

$$l = \sqrt{x^2 + 25} + \sqrt{\left(\frac{25}{x}\right)^2 + 25}.$$

Perfectly true, but perhaps unexpected. It is unusual in high school algebra for the sum of two such differently looking square roots to equal yet another different single square root. You can see why it is true in our model, but why is it true algebraically?

The question about going around the 120° corner leads to another interesting problem. If we have an obtuse triangle with sides of length a and b on either side of the 120° angle, how long is the side opposite the 120° angle? By the law of cosines, we get that $c^2 = a^2 - 2ab \cos 120^\circ + b^2 = a^2 + ab + b^2$. So it is natural to ask the question, "What corresponds to Pythagorean triples in a 120° triangle?" Are there integers a , b , and c such that $a^2 + ab + b^2 = c^2$? Well, $a = 5$ and $b = 3$ yield $c = 7$, so it can certainly happen. Other examples are $(7, 8, 13)$ and $(7, 33, 37)$. [No, it is not true that all solutions involve 7. There is $(5, 16, 19)$.] Here is a general formula for solutions: pick non-negative integers m and n , and let

$$a = 3n^2 + 2mn$$

$$b = 2mn + n^2$$

$$c = 3n^2 + 3mn + n^2.$$

See the following reference for an application of this bit of mathematics in the context of high-speed photography.

Reference

Gilbert, E.N. (1963)., Masks to pack circles densely, *J.SMPTE* 72, 606-608

TALE OF THE TAPE

Teacher's Guide — Getting Started

Elizabeth Brennan & Andrew Sanfratello
The Hewitt School & Mohegan Lake, NY

Purpose

In this two-day lesson, students will model the path of a baseball in flight and use that model to determine how far the ball will travel (in ground distance). Students then use those ideas to apply them to skeet shooting where they determine not just the flight of the clay disk, but also the flight of the pellet and their intersection point.

Ideally, the lesson will involve solving a system of linear equations to determine the function and solving a quadratic equation to find the roots of this function, although other models are encouraged. For some examples, you may be able to factor to solve the resulting quadratic equation, but if the polynomial is prime, the quadratic formula, completing the square, or the graphing calculator can be used.

Prerequisites

Knowledge of quadratic functions, solving systems of linear equations, and solving quadratic equations is required. Students should also know the meaning of domain and range of functions.

Materials

Required: A tennis ball or other round object that can be tossed by students.

Suggested: Graphing calculators, geometry software (for diagrams), and a SMART Board.

Optional: None.

Worksheet 1 Guide

The first three pages of the lesson constitute the first day's work. Students work towards determining a good model that follows the path of the baseball. They are encouraged to focus on what variables are important to consider as well as the domain of the function. Students are then given measurements for a hit baseball to apply their model to the real world of baseball.

Worksheet 2 Guide

The fourth and fifth pages of the lesson constitute the second day's work, which focuses on skeet shooting. Students use their models from the baseball example to model the flight of the clay target while also modeling the path of a pellet to determine how far the pellet traveled to hit the clay target.

CCSSM Addressed

A-CED.1: Create equations and inequalities in one variable and use them to solve problems.

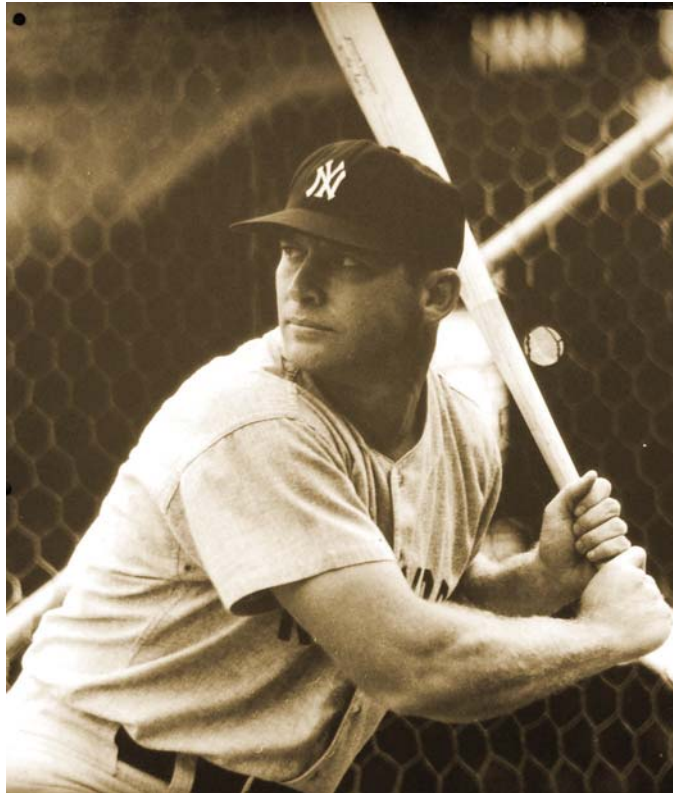
A-REI.11: Explain why the x -coordinates of the points where the graphs of the equations $y = f(x)$ and $y = g(x)$ intersect are the solutions of the equation $f(x) = g(x)$; find the solutions approximately, e.g., using technology to graph the functions, make tables of values, or find successive approximations. Include cases where $f(x)$ and/or $g(x)$ are linear, polynomial, rational, absolute value, exponential, and logarithmic functions.

F-IF.5: Relate the domain of a function to its graph and, where applicable, to the quantitative relationship it describes.

TALE OF THE TAPE

Student Name: _____ Date: _____

You are sitting in the stands at a baseball game when a towering home run goes over your head and you cannot help but wonder how far the ball will go if unimpeded. The “long ball” is considered by many to be one of the most exciting parts of the game. One of baseball’s great home run hitters, Mickey Mantle, was said once to have hit a ball a distance over 700 feet! Is there an accurate way to estimate the distance a home run hit could travel if unimpeded?



Public domain Photo—www.flickr.com/photos/tonythemisfit

Leading Question

How can you determine how far the ball could travel if unimpeded?

TALE OF THE TAPE

Student Name: _____ Date: _____

1. What mathematical properties does the path of the baseball have?

Try throwing a ball in the air. Are there any symmetries in its flight?

2. What information is essential in order to create a model for the flight of the baseball? What aspects of the flight of the baseball might be too difficult to use in your model? What information is less important to your model?

3. What type of mathematical model can you use to represent the path of the ball? Create your model.

TALE OF THE TAPE

Student Name: _____ Date: _____

4. A baseball player makes contact with a ball 4 feet above the ground. The ball reaches its maximum height of 84 feet when the ball is 200 feet from home plate, measured along the ground. The ball hits the scoreboard at a height of 64 feet when the ball is 300 feet from home plate. Use your model to find where it would land if it were not impeded.

What mathematical value describes the landing point of the baseball?

5. If you obtained more than one numerical value, how do you know which one is the solution to your problem? If you have more than one value, what do these values mean? What does the presence of one negative and one positive value each say about the domain of your function? What would be an appropriate domain for your function?

6. How could you solve this problem if you were given only the maximum point and the initial point, and not the impact point with the stands?

What special properties of the maximum can you use to help you?

TALE OF THE TAPE

Student Name: _____ Date: _____

While the path of a baseball uses only one mathematical function, other sports sometimes require more than one function to be used in the model. In skeet shooting, a clay disk is launched into the air and the goal is to shoot the disk in mid-flight.

7. What function might you use to model the flight of the disk? How much (or little) information do you need to create a model that traces its path?

8. What type of function might you use as a model to approximate the path of a pellet that hits the clay disk? What information do you need to create a model that traces its path?

9. How can you use the models of the two paths to indicate where the pellet hits the clay disk?

TALE OF THE TAPE

Teacher's Guide — Possible Solutions

The solutions shown represent only some possible solution methods. Please evaluate students' solution methods on the basis of mathematical validity.

1. The flight of the baseball is uniformly continuous (i.e., smooth) and its path follows that of a parabolic curve.
2. Instantaneous points during the flight of the ball are essential for creating an accurate model. Gravity, friction, and the spin on the baseball are a few of the things that do act upon the baseball but are too difficult to consider without advanced mathematics or physics. A variable such as speed has little importance in creating the model.
3. While the path of the ball is parabolic in nature, some students might choose to model with other functions such as arcs of circles or trigonometric functions.
4. For a parabolic model, the equation is $y = (-1/500)x^2 + (4/5)x + 4$. This has a positive solution of approximately 405, so the ball would land about 405 feet from home plate.
5. For a “concave down” parabola whose y -intercept is positive, there will always be two real solutions, one positive and one negative — the positive solution is the correct one to use in this case. Negative distance does not make sense in this context. The domain of the function should be restricted to non-negative values.
6. In the parabolic model, the x -coordinate of the maximum is equal to $-b/(2a)$. Since $c = 4$ (from the initial height), using the formula for the x -coordinate of the vertex gives $200 = \frac{-b}{2a}$. It follows that $b = -400a$ and substituting results in $a = \frac{-1}{500}$ and $b = (-400) \left(\frac{-1}{500} \right) = \frac{4}{5}$.
7. The flight of the clay disk is comparable to the flight of a baseball. It can be modeled with a parabola in a similar way as the flight of a baseball.
8. While a pellet's flight is also parabolic, a good approximation can be made with a linear function over short distances. Two points are necessary to determine the formula for a linear function.
9. The point of impact is the intersection point of the two functions. This can be calculated by setting the two functions equal to each other and solving.
10. A good model will have the clay disk's path modeled as a parabolic function with a negative value for the x^2 coefficient. The pellet's path will be modeled with a linear function and the two should have at least one point of intersection.
11. One function that models the flight of the clay disk is $f(x) = -x^2 + 4x + 5 = -(x+1)(x-5)$. The point (3, 8) lies on this parabola and represents where a linear approximation of the pellet's path would hit the disk. Using the distance formula or the Pythagorean Theorem, students can calculate that the approximate distance that the pellet traveled is $\sqrt{73} \approx 8.5$ yards. If you know the approximate distance the pellet traveled, and you know the speed of the clay disk and the pellet, you can determine when to fire the gun.
12. Answers will vary depending on what ideas students have and what they discover while researching.

TALE OF THE TAPE

Teacher's Guide — Extending the Model

The most common models for projectile motions ignore air resistance, and the only force acting on the projectile is taken to be gravity. The mathematics involved is that of parabolas and there are lots of pretty problems. For example, how high should the roof of a domed baseball stadium be?

The truth of the matter is that there is a force opposing the motion. When mathematics or classical physics models try to take this into account, the most common method takes the force opposing the motion to be proportional to the square of the velocity. Are either of these true? Well, at relatively low velocities, the force proportional to velocity is said to be pretty accurate; at high velocities, the force proportional to the square of the velocity is endorsed. For the specific application to baseball, there is no better source of information than Robert K. Adair's book *The Physics of Baseball*. This tells you that the real world is pretty complicated, but in the region of speeds most relevant to baseball, Adair prefers the "square of the velocity" model. One might harbor some suspicions that the true power of the velocity can vary a good deal.

Not that truth is the only influence on the design of a mathematical model of motion with air resistance. Mathematical analysis applied to motion with no air resistance leads to parabolic models, and a minor difficulty is the pedagogic confusion between height as a function of time and height as a function of horizontal distance traveled. Both of them are quadratics, you see, and the possible confusion between y as a function of t and y as a function of x can be quite troublesome because both are parabolas!

If you wish to model projectile motion with a force opposing the motion proportional to some power b of velocity, only the cases $b = 1$ and $b = 2$ have analytic solutions, although the solution with $b = 1$ is easier than the solution with $b = 2$. For other constant values of b , or, even worse, values of b possibly varying with velocity and air density, your friendly numerical analyst or computational physicist will take over, and the mathematical analyst will mourn the loss of elegance.

Let us take a quick look at the case where $b = 1$, in the simplest scenario of a ball going straight up and then down. The equation of motion on the way up is $\frac{d^2y}{dt^2} = -g - c\left(\frac{dy}{dt}\right)$, where $c > 0$. Here, $\frac{dy}{dt}$ is positive, and the air resistance acts with the force of gravity to slow the motion. You can write this as

$$\frac{d}{dt}\left(\frac{dy}{dt}\right) = -g - c\left(\frac{dy}{dt}\right)$$

or

$$\frac{\frac{d}{dt}\left(\frac{dy}{dt}\right)}{\frac{g}{c} + \frac{dy}{dt}} = -c.$$

When you integrate this, you obtain

$$\ln\left(\frac{dy}{dt} + \frac{g}{c}\right) = -ct + \ln k,$$

which implies

$$\frac{dy}{dt} = ke^{-ct} - \frac{g}{c}.$$

TALE OF THE TAPE

Teacher's Guide — Extending the Model

The constant of integration, k , turns out to be $v_0 + \frac{g}{c}$, where v_0 is the initial velocity at $t = 0$.

You can integrate this once more, and remembering that $y(0) = 0$, you obtain

$$y(t) = \left(v_0 + \frac{g}{c}\right) \left(\frac{1}{c}\right) (1 - e^{-ct}) - \frac{g}{c} t.$$

It is instructive to see that when c goes to 0, the usual equation of motion results. Also, you may find the value of t at which y is maximized, and the corresponding maximum value of y . Calculator plots of the results with varying c give you insight into the size of the effect of the air resistance. From an incorrigible analyst, one more comment. The equation when $b = 2$ succumbs to a corresponding analytic attack, except that the first integration leads to an arctangent rather than a natural logarithm. It is, admittedly, messier.

So, do the books emphasize the cases $b = 1$ and 2 because a traditional mathematical analysis is possible, or because they are reasonable models of some aspect of physical truth? As Alfred Doolittle says in "Pygmalion" when Henry Higgins asks him whether he is an honest man or a rogue, "a little of both, Henry, like the rest of us: a little of both."

References

- Adair, R.K. (2002). *The physics of baseball*. New York: Perennial.
Shaw, G.B. (1912). *Pygmalion: A play in five acts*.

UNSTABLE TABLE

Heather Gould
Stone Ridge, NY

Teacher's Guide — Getting Started

Purpose

Have you ever tried to eat on an unstable, tippy table? No doubt drinks and soup were spilled easily! Restaurant wait staff often fold paper napkins to wedge under one of the legs to stabilize the table.

In this two-day lesson, students learn to stabilize a table without the use of napkins — they can rotate it up to 90° . The result is counterintuitive but can be verified mathematically.

Prerequisites

Knowledge of slope and continuous functions.

Materials

Required: Small furniture such as doll furniture, construction paper, scissors, and string.

Suggested: None.

Optional: None.

Worksheet 1 Guide

The first four pages of the lesson constitute the first day's work. Students are encouraged to experiment with small furniture to check to see if they can stabilize it by a rotation in various spots around the classroom. Students develop a model in two dimensions that will help them understand the situation more completely. Students experiment with the model and find the commonalities between the two- and three-dimensional worlds. Finally, they begin to build an intuitive understanding of the Intermediate Value Theorem.

Worksheet 2 Guide

The fifth through eighth pages of the lesson constitute the second day's work. Students continue to work with the two-dimensional model, but the situation becomes more complicated — it is the two-dimensional version of a 4-legged table in three dimensions. They find through experimentation that it always is possible to stabilize a 3-legged table in two dimensions and give a mathematical explanation that relies on the Intermediate Value Theorem. Finally, they extend their model to the situation at hand (a 4-legged table in three dimensions) and mathematically show that it always is possible to stabilize the 4-legged table.

CCSSM Addressed

A-CED.1: Create equations in one variable and solve them.

F-IF.4: For a function that models a relationship between two quantities, interpret key features of graphs and tables in terms of quantities, and sketch graphs showing key features given a verbal description of the relationship.

F-BF.4: Write a function that describes a relationship between two quantities.

F-LE.5: Interpret the parameters of a linear function in terms of context.

UNSTABLE TABLE

Student Name: _____ Date: _____

Have you ever tried to eat a bowl of soup on an unstable, wobbly table? What happened? If you were in a restaurant, a waiter may have wedged a folded paper napkin under one of the table's legs to stabilize it — but there's another way! This is because the problem usually isn't with the table's legs; the problem is that the floor is uneven!



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Leading Question

How can a restaurant's wait staff use the unevenness of the floor to help them stabilize an unstable table?

UNSTABLE TABLE

Student Name: _____ Date: _____

1. It seems that most of the instability in tables is caused by uneven floors. Experiment by placing furniture with 3 or 4 legs in different places around the classroom. Is your furniture unstable? If so, try rotating it little by little. Does it become stable? Repeat this experiment several times in different spots around the classroom. Fill in the table below.

Trial #	Degree of Rotation Needed to Stabilize the Table
1	
2	
3	

2. Do you think some rotation will always cause the table to become stable? Why or why not?

3. If you were unable to stabilize the table, it could be that one leg is shorter than the others. Stretch string between the tips each pair of opposite legs. How can you tell if the tips of the legs are coplanar?

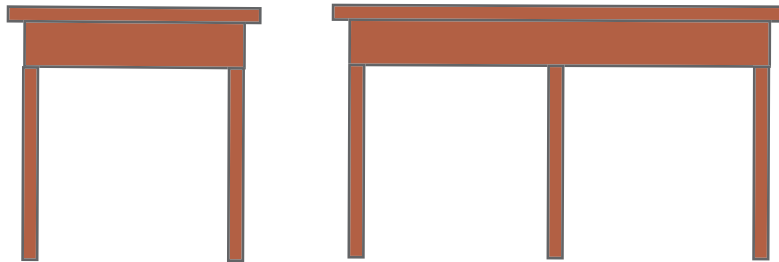
*What does "coplanar" mean?
What do you know about
things that are coplanar?*

UNSTABLE TABLE

Student Name: _____ Date: _____

If the tips of the table are coplanar, it will be stable when the floor is level. If you conduct more trials by rotating a table on an uneven floor, you should observe that rotation always seems to stabilize the table — but to show that it is true requires a mathematical model. Sometimes, to get started, it helps to model a *similar but simpler situation*.

Two-dimensional objects are usually simpler to study than three-dimensional ones. Even though the two-dimensional tables aren't useful in the real world, they may be helpful in the mathematical world. In the two-dimensional world, 2- and 3-legged tables would look like the pictures below.



4. What should represent an uneven floor in the two-dimensional world? Use construction paper to cut out an uneven two-dimensional floor and several two-dimensional tables.

What properties might an uneven floor have? Would it keep rising forever or would it rise and fall and stay around the same height?

5. In a two-dimensional model, a rotation in three dimensions must be replaced by a “slide.” Slide a 2-legged two-dimensional table along the two-dimensional floor until both legs contact the floor. Try this for several different starting positions. Is it difficult to stabilize the table? Explain your findings.

UNSTABLE TABLE

Student Name: _____ Date: _____

6. Do you observe a common property between a 2-legged two-dimensional table and a 3-legged three-dimensional table? Explain your thoughts.

7. What do you observe about changes in the slope of the top of the 2-legged table as it slides along an uneven floor?

8. If the slope of the tabletop is positive at one point and negative at another, what must happen in between? Explain what this tells you about the tabletop.

UNSTABLE TABLE

Student Name: _____ Date: _____

9. Consider the 3-legged two-dimensional table on the uneven two-dimensional floor. Slide it until all 3 legs contact the floor and record the length of the slide needed to stabilize the table. Repeat this experiment several times starting at different places on the floor. Record your results.

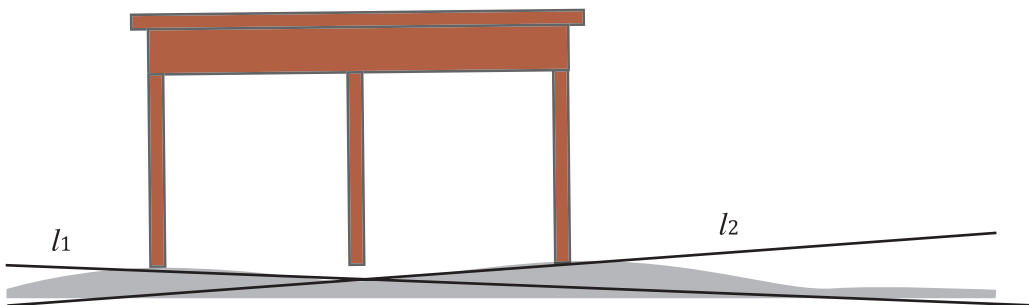
Consider the length of the slide in terms of the distance between adjacent legs.

Trial #	Length of Slide Needed to Stabilize the Table
1	
2	
3	

10. Was it always possible to stabilize the 3-legged table on the uneven two-dimensional floor? Explain.

11. What do your trials indicate about the length of the slide required?

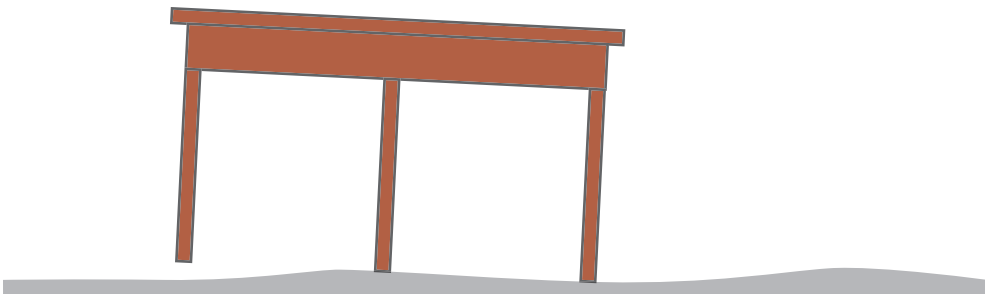
12. Consider two slope functions: l_1 , the slope of the line from the first leg to the floor at a point below the second leg, and l_2 , the slope of the line from the third leg to the floor below the second leg. An example is shown below. In the example, the slope of l_1 is negative. Is the slope of l_2 positive or negative? Explain.



UNSTABLE TABLE

Student Name: _____ Date: _____

13. Let S_1 and S_2 be slope functions that have different values as the table slides along the floor. Subtract these two functions to obtain $S = S_1 - S_2$. Is S a continuous function? Explain.
14. If S is continuous, what must occur between points where $S > 0$ and $S < 0$? Explain.
15. At the point where $S = 0$, what must be true of S_1 and S_2 ? What does that tell you about the position of the middle leg with respect to the uneven floor?
16. Suppose the first leg of the table is above the floor while the two other legs touch the floor, as shown below. What slopes would you use to show that as the two-dimensional table slides along the floor, at some position all three legs will touch?



UNSTABLE TABLE

Student Name: _____ Date: _____

If you have understood how two-dimensional tables slide along an uneven two-dimensional floor, you should be able to extend the two-dimensional model to three dimensions. Begin by thinking of the legs of a 4-legged three-dimensional table as the table is rotated on the uneven floor. Actually, if a two-dimensional “floor” is bent to form a circle, it’s just like the arc around which a three-dimensional table rotates.

17. Will a 4-legged three-dimensional table always have 3 of its legs touching the uneven floor? Explain.
18. Can a continuous function be found that is positive somewhere and negative somewhere else? If so, what would that tell you about the function?
19. Experiment! Perhaps two or more slope functions will suffice. Since 3 legs of an unstable 4-legged table always will touch the floor, exactly 1 leg always will be above the floor, say, by k mm. Connect the opposite legs of the 4-legged table that do touch the floor with a line segment, l_1 . At each end, the height above the floor is 0 mm. To create l_2 , connect the third leg with the point on the floor below the fourth leg (the one that doesn’t touch the uneven floor). The slope of line l_1 is 0 as is the slope of line $l_2 - k$, that is, $S_1 = 0$ and $S_2 = -k$. Subtract these two functions to obtain $S = S_1 - S_2$. Of course, the values of S_1 , S_2 , and S change as the table is rotated. How do the values of S_1 and S_2 change when the table is rotated exactly 90° ?

UNSTABLE TABLE

Student Name: _____ Date: _____

20. Considering what happens to S_1 and S_2 when the 4-legged table is rotated by exactly 90° , what must happen to S in between? What does this mean in terms of the table?

21. What can you say about the possibility of stabilizing a 4-legged table on an uneven floor? Are you surprised by what your model shows?

UNSTABLE TABLE

Teacher's Guide — Possible Solutions

The solutions shown represent only some possible solution methods. Please evaluate students' solution methods on the basis of mathematical validity.

1. Answers will vary. However, 3-legged tables should always be stable and never need to be rotated because any 3 points define a plane; 4-legged tables should never require more than a 90° rotation.
2. Most students will believe that it is not always possible to stabilize a table with a rotation. Contrary to what students believe, a rotation always will stabilize a table with legs whose ends are coplanar on a surface that is not always increasing or always decreasing.
3. The tips of the legs are coplanar if the strings, when pulled taut, do not bend.
4. Below is a sketch of a possible uneven two-dimensional floor. The floor will rise and fall a bit, but it will generally stay around the same height.



5. There should be no difficulty stabilizing a 2-legged table in two dimensions. It should be stable in any position it is placed.
6. The feet of a 2-legged table always define a line (as any 2 points define a line). The feet of a 3-legged table always define a plane. The concept of a line in two dimensions is similar to the concept of a plane in three dimensions.
7. The slope of the tabletop will change from negative, to 0, to positive, to 0, to negative, and so on as long as it keeps sliding.
8. The slope must be 0 at some point in between. This means that the table eventually will not only be stable, but will also be level.
9. Answers will vary. The length of the slide never should be longer than the distance between adjacent (consecutive) legs.
10. It is always possible to stabilize the table and, in fact, it always can be done with a slide whose length is less than or equal to the distance between adjacent legs.
11. The slide never was longer than the distance between adjacent legs.
12. The slope of l_2 is positive. Unless they are both 0, the slopes of l_1 and l_2 will always have opposite signs.
13. Yes, S is a continuous function since both l_1 and l_2 are continuous functions and subtraction is a continuous operation.
14. Since S is continuous, it must be 0 at some point in between.
15. If $S = 0$, then $S_1 = S_2$ and the middle leg must be touching the floor — the table will be stable.
16. The slopes of l_1 and l_2 still are used. In the picture, one must find l_1 by “wobbling” the table so that the first leg is touching the floor. Thus, S_1 is positive and S_2 is negative.
17. Yes, because any 3 points define a plane.
18. Yes, one can. Define lines on opposite legs of the 4-legged three-dimensional table. This means that the slope would be 0 somewhere in between and the table would be stabilized.
19. At 90° , the slopes S_1 and S_2 exchange their previous values. So, if S_1 was 0, it would become $-k$ and if S_2 was $-k$, it would become 0.
20. Since the values of S_1 and S_2 exchanged values, then S changed from k to $-k$. It must have been 0 in between. Thus, the table can be stabilized within a 90° rotation.
21. It always is possible to stabilize a 4-legged three-dimensional table. This result is usually surprising.

UNSTABLE TABLE

Teacher's Guide — Extending the Model

Please keep in mind that “stabilize” can have two different interpretations. One interpretation is that all the legs of the table are on the floor at the same time so that it doesn't take somebody's foot to hold the table down or a napkin stuffed under a short leg. Another is that the tabletop is also horizontal so that nothing will slide off of it. Generally speaking, the first interpretation tends to apply to the three-dimensional table, and the second to the two-dimensional table.

We want to take a more careful look — you might even say “rigorous” look — at the mathematics underlying the simplest form of this modeling problem. Let us assume that the floor covers the interval $[0, 1]$ and that the height of the floor is given by a continuous function $h(x)$. We assume that $h(0) = h(1)$. Let the table have length $1/2$. Does it follow that there must be an $x \in [0, 1]$ such that $h(x + 1/2) = h(x)$? That would be a stable position of the table. It does follow, and the proof is given below.

Proof: Let $g(x) = h(x + 1/2) - h(x)$, which is defined for $x \in [0, 1/2]$. Either $g(0) = 0$ or it doesn't. If $g(0) = 0$, then $x = 0$ is a value of x with the desired property. If $g(0) \neq 0$, then we may assume without loss of generality that $g(0) > 0$. Then we claim that $g(1/2) < 0$. Why? Well, $g(0) + g(1/2) = h(1/2) - h(0) + h(1) - h(1/2) = h(1) - h(0) = 0$, and so if $g(0) > 0$, then $g(1/2) < 0$. But $g(x)$ is a continuous function because $h(x)$ is continuous. Hence by the Intermediate Value Theorem, there is a value of $x_0 \in (0, 1/2)$ such that $g(x_0) = 0$. By definition of g , $h(x_0 + 1/2) = h(x_0)$.

A very similar argument will work for a table of length $1/3$. We set $g(x) = h(x + 1/3) - h(x)$. Then $g(0) + g(1/3) + g(2/3) = 0$, and if $g(0) > 0$, then at least one of $g(1/3)$ and $g(2/3)$ must be negative. Therefore $g(x) = 0$ somewhere in $[0, 2/3]$. The same argument will work for a table of length $1/n$, where n is an integer.

The result is false for a table of length α if $\alpha > 1/2$. For example, let $h(x) = x$ in the interval $(0, 1 - \alpha)$, $h(x) = (x - 1)$ from $1 - \alpha$ to 1 , and continuous in the middle.

Question: What happens if $\alpha = 2/5$, or any rational number less than $1/2$ and not of the form $1/n$? Does there have to be an x such that $h(x + 2/5) = h(x)$? No, there doesn't! And there cannot be. For the proof of this see “A Stable One-Dimensional Table” in *Consortium*.

Reference: Pollak, H. O. (2011, Fall/Winter). A Stable One-Dimensional Table. *Consortium*, 101, 15-16.

SUNKEN TREASURE

Benjamin Dickman
Brookline, MA

Teacher's Guide — Getting Started

Purpose

In this two-day lesson, students help the crew of a shipwreck recovery team minimize the amount of work done to remove treasure chests from a ship lost at sea. The divers must move the chests to a rope that is between their locations coming from the recovery team's boat above. The captain of the boat's crew insists on placing the rope in one spot; he doesn't want to waste time and money moving it each time a chest is collected.

Prerequisites

An understanding of basic algebra and geometry with triangles are needed.

Materials

Required: Large, flat pieces of cardboard, string, small weights, and scissors (to pierce cardboard).

Suggested: Rulers or straightedges, compasses, geometry software, washers (to place on holes in cardboard to reduce friction).

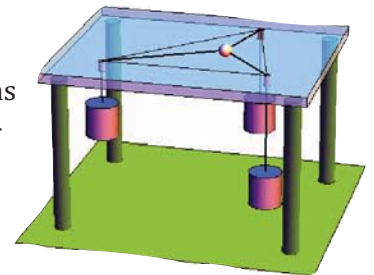
Optional: None.

Worksheet 1 Guide

The first three pages of the lesson constitute the first day's work. A physical model of the situation can be constructed in the classroom from cardboard. To do this, cut two holes in the cardboard 40 cm apart from each other and thread two strings through the holes. Tie them together above the cardboard so that the lengths of the strings below the knot are equal. Set the cardboard between two posts or two tables so that it is level and the weights can hang freely. Students should experiment by using equal and unequal weights at the end of each string. The position of the knot should help determine where to position the rope from the boat.

Worksheet 2 Guide

The fourth and fifth pages of the lesson constitute the second day's work. The cardboard model above should be modified to fit the "three chests" problem (shown at right). Students will need to experiment using different combinations of weights: all three the same, two the same and one different, and all three different. Students learn the definition of **work** and will modify their ideas about how work should be defined from the first day to use this mathematical definition.



CCSSM Addressed

A-CED.1: Create equations and inequalities in one variable and use them to solve problems.

G-MG.3: Apply geometric methods to solve design problems.

SUNKEN TREASURE

Student Name: _____ Date: _____

A shipwreck containing treasure chests filled with gold and silver was discovered recently in the Bermuda Triangle. Underwater photos revealed two treasure chests spaced 40 meters apart and it is up to you to determine how best to retrieve them.

Your boat has a rope that can be lowered and tied around the treasure chests, but your captain insists he doesn't want to sail back and forth all day. He says that you have to choose one place to lower the rope, and then you can swim down with it. To collect the treasure, the chests must be moved to the end of the rope to be lifted to the surface.



Undersea Photo: © Kirill Zelianodjevo | Dreamstime.com

Chest: © Johanna Goodyear | Dreamstime.com

Leading Question

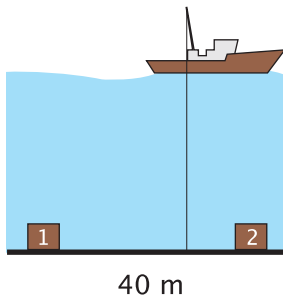
What is the best location to place the recovery ship and drop the rope so you don't upset the captain?

SUNKEN TREASURE

Student Name: _____ Date: _____

1. What information is necessary to have before a mathematical model is constructed? What variables do you have to consider? What variables should be not be taken into account?

2. Call the treasure chest on the left Chest 1 and the treasure chest on the right Chest 2. If you know the distance from Chest 1 to the rope, how can you express the distance from Chest 2 to the rope?



SUNKEN TREASURE

Student Name: _____ Date: _____

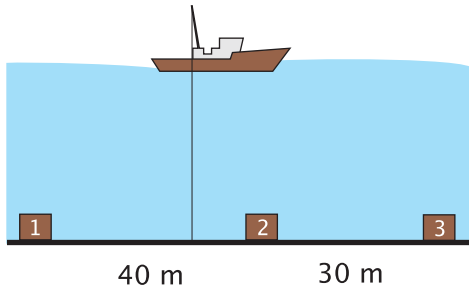
3. When you dive down, you will have to move the chests over to the rope. What's a good way to measure the amount of work done to move the chests? What variables should be taken into account in this measurement? Is there a way to consider all of these variables together?

4. You estimate that each of the chests has the same weight. You want the total work you do to move both chests to be as little as possible. Where should you lower the rope? How would your answer change if the chests weighed different amounts? Use a cardboard model to experiment and test your ideas.

SUNKEN TREASURE

Student Name: _____ Date: _____

5. When you dive to the bottom, you find that there is a third treasure chest! Fortunately, they all are lined up in a row. If you still want to minimize the work in moving all the chests, where do you place the rope now assuming that they each weigh the same? Provide a mathematical explanation for your reasoning.

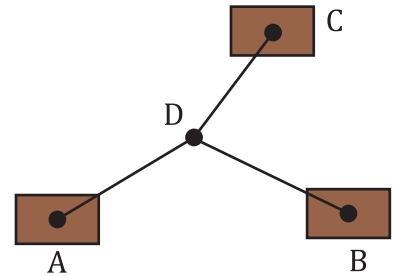


SUNKEN TREASURE

Student Name: _____ Date: _____

In mathematics and physics, **work** has a very precise meaning. **Work** is the amount of energy transferred by a force acting over a certain distance. Here, “energy transferred by a force” means the same thing as “weight.” The equation is: $\text{Work} = \text{Force} \cdot \text{Distance}$.

6. You did so well on your first dive that your captain is bringing you to another site. This site has 3 treasure chests, all equal weights, but they don't lie along a line. How do you minimize the amount of work done to move the chests to the rope? Use a cardboard model to find point D .



7. If the chests in question 6 did not all have the same weight, how would the model change? Modify the cardboard model for this physical situation. What happens? Can you give a mathematical explanation for what is going on?
8. What are the differences between a physical model and a mathematical model? What are some advantages and disadvantages of each?

SUNKEN TREASURE

Teacher's Guide — Possible Solutions

The solutions shown represent only some possible solution methods. Please evaluate students' solution methods on the basis of mathematical validity.

1. The weight of the chests and the distance between the chests and the rope are the two variables that need to be considered. On the other hand, the length of the rope and the depth are not variables that need to be considered in this model.
2. If the distance between Chest 1 and the rope is x meters, then the distance between the rope and Chest 2 is $(40 - x)$ meters.
3. **Work = Force • Distance.** In this example, weight is an appropriate substitution for force. You could also measure the amount of work done in time, energy expended by the divers, or even cost of the entire operation.
4. When the chests weigh the same, it doesn't matter where the rope is positioned, as long as it is between the two chests. When the weights are varied, it is most efficient to place the rope directly above the heavier of the two chests.
5. Placing the rope over the middle chest will result in the smallest amount of work. Explanations will vary but they all may be valid if they confirm the correct placement.
6. Three points that are not collinear will create a triangle. The point D , for which the sum of the distances to the vertices is least, is called the Fermat point. The angle formed in the interior of the triangle by D and any two of the three chests is 120° .
7. The weights on the model would need to be adjusted accordingly. The knot would be pulled towards the heaviest weight and shifted near the second heaviest weight. As a possible extension, you may want to try using one set of three fixed weights and see what occurs, although note that each combination of weights will have a unique point.
8. Physical models often do not run as smoothly as a mathematical model would suggest. There may be variables that were not considered in the mathematical model for simplicity's sake that can greatly affect the outcome of the physical model. Each have their own upsides and pitfalls.

SUNKEN TREASURE

Teacher's Guide — Extending the Model

You may wish to consider the extension to two dimensions with three locations. You would expect that if the three locations were close to being in a straight line that the solution would look similar to the one-dimensional case.

First look at the problem if the three weights w_1 , w_2 , and w_3 are equal, and are located at points P_1 , P_2 , and P_3 , respectively. If the triangle formed by the P_i 's is sufficiently obtuse, then the optimal location for the rope is at the vertex of the obtuse angle. This is true, as long as the obtuse angle is at least 120° . If all angles are less than 120° , the optimum location for the boom is the point P inside the triangle at which PP_1 , PP_2 , and PP_3 meet at 120° . There are a number of nice geometric proofs of this, but the easiest one is by physics. In order to see this, we might as well assume that the weights at the three locations are general rather than equal.

Imagine a piece of Plexiglass, or a sheet of wood, and drill holes at the three locations of P_1 , P_2 , and P_3 . Tie three pieces of string together at a point P , and run the three strings — one through each hole — and attach the weights, w_i , to the strings of equal length going through their respective P_i . Let the configuration go. It should settle into a configuration of minimum potential energy, and it follows with a little energy argument that this will minimize the sum of the three products w_1 , w_2 , and w_3 multiplied by their respective distances PP_1 , PP_2 , and PP_3 .

If one of the three w_i 's is much more than the sum of the others, then P will be pulled to P_i , and you get the same end point problem as before.

There is a geometric construction for the general 3-point case which uses Ptolemy's Theorem.

If you have more than three points, locating a single point P that minimizes the sum of the distances is a classic problem for which there is literature, but nothing especially simple.

If you have four points and you want the shortest network connecting them, that's a different problem. The literature about this problem goes back to Gauss, who put the solution into a letter to Schumacher but didn't publish it. Gauss became interested in it because his son was working for the Duchy of Hanover, planning its first railroad. There was earlier interest in the problem for planning canals in England. It is not the same problem unless $n = 3$.

ESTIMATING TEMPERATURES

Heather Gould
Stone Ridge, NY

Teacher's Guide — Getting Started

Purpose

In this two-day lesson, students will model temperature data. They will use “known temperature stations” in order to estimate temperatures at any given point accurately. Websites that give the temperature at a specific place typically do not give the actual values; they give an estimate based on meteorological data.

Explain to students that temperatures are not measured everywhere and educated estimates need to be made. Have the students imagine they are meteorologists interested in making a model to estimate temperature at a given time and at a given location.

Prerequisites

Students need to understand ratios and equations in one variable, as the lesson is heavily dependent on these areas. Additionally, reading, interpreting, and understanding graphs is important for completing the lesson.

Materials

Required: Rulers or straightedges.

Suggested: Graphing paper or a graphing utility.

Optional: (For three-dimensional models) Cardboard, sticks or drinking straws, and scissors.

Worksheet 1 Guide

The first three pages of the lesson constitute the first day's work. Students are asked to estimate the temperature at a point on a map between two other points where the temperature has been measured. It is important that students understand that the diagrams given are drawn to scale. This fact should arise from discussion about variable identification in questions 1 and 2. The students should begin to formulate ideas about linearity. Questions 4 and 5 ask students to extend their model when the unknown points do not fall in a straight line with two known temperature stations. There will be a variety of solution methods, but each should use the concept of linearity or a constant rate of change between two points.

Worksheet 2 Guide

The fourth and fifth pages of the lesson constitute the second day's work. Students are first given a definition of a linear function and then questions have students making connections between their Day 1 models and the graph of a linear function. Then students will give their description of the meaning of average rate of change and its relation to linear functions. They will be challenged to calculate the rate of change of a linear function.

CCSSM Addressed

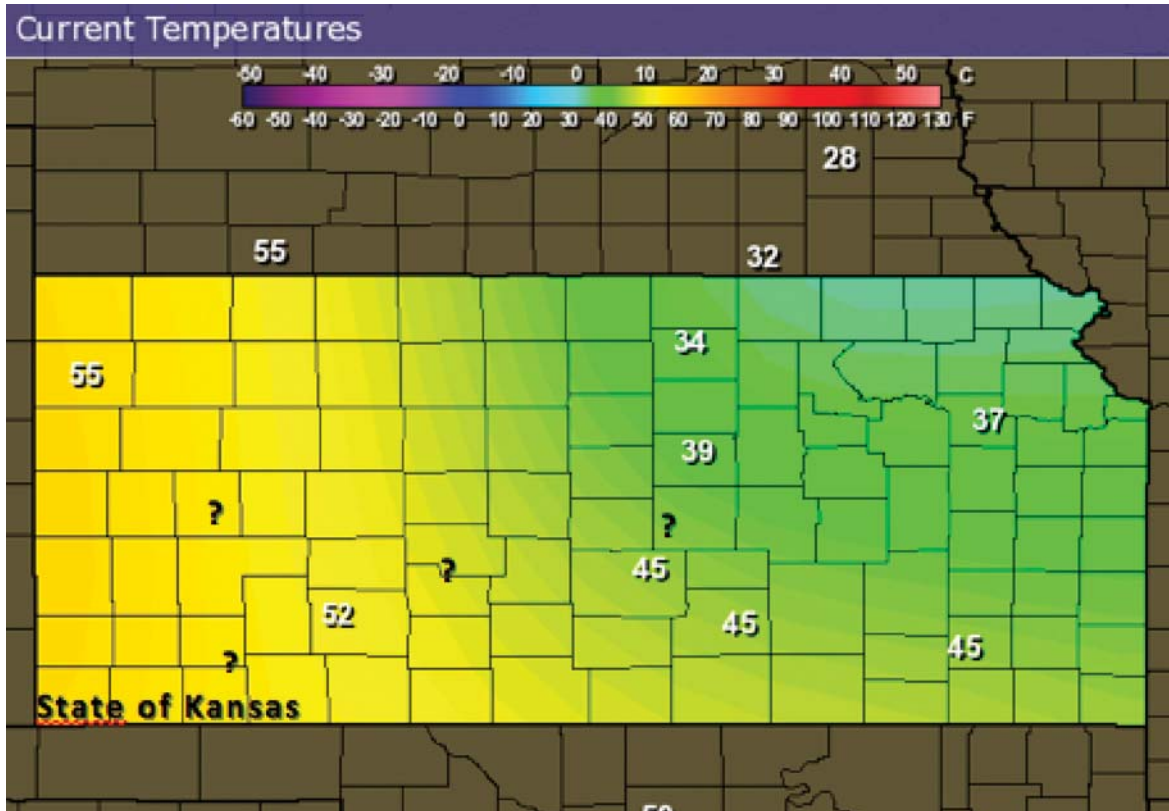
A-CED.3: Represent constraints by equations or inequalities, and by systems of equations and/or inequalities, and interpret solutions as viable or non-viable options in a modeling context.

F-IF.6: Calculate and interpret the average rate of change of a function (presented symbolically or as a table) over a specified interval. Estimate the rate of change from a graph.

ESTIMATING TEMPERATURES

Student Name: _____ Date: _____

When you look on websites such as www.weather.com to find out the current temperature, you usually don't get the actual measured temperature for your town — it's an educated estimate!



© The Weather Channel

Leading Question

How would you create a model like one a meteorologist would use to estimate the temperature?

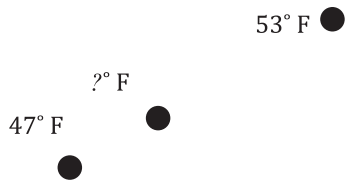
ESTIMATING TEMPERATURES

Student Name: _____ Date: _____

1. How would you expect temperatures to change in between two towns? Use your experience to make an educated guess. Would you expect the temperature to change gradually or suddenly? Explain.



2. How can you estimate the change in temperature between two towns? Use your ideas from above to estimate the temperature in between those towns with known temperature. Show your work.

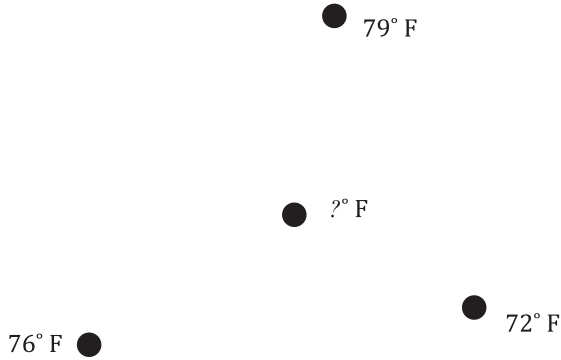


3. Describe a mathematical model for estimating the temperature in a given town between two towns for which the temperatures are known. Write your description in words first, then in mathematical symbols.

ESTIMATING TEMPERATURES

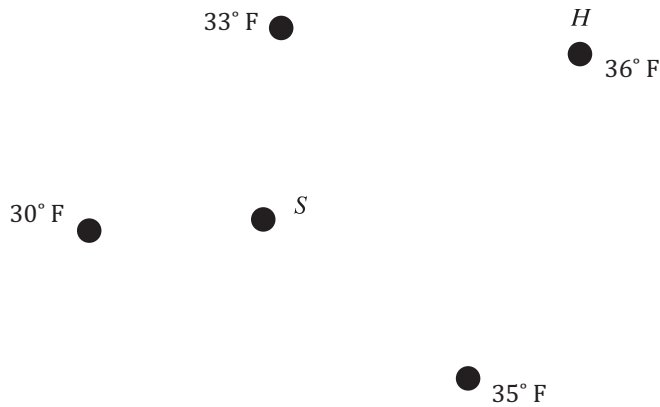
Student Name: _____ Date: _____

4. How would you estimate the temperature in a town that isn't between two towns where the actual temperatures are measured?



Can you still use the model from before? Do you need to modify it or do you need to make a brand new one?

5. It is a cold, rainy day. You and your friends want to drive to an indoor skate park (*S*) from home (*H*). Your parents are worried that it will get colder as you get closer to the skate park and the rain will freeze; you're not allowed to drive if there's a chance of sleet or snow. Use the map below to determine if it's safe to go.



6. Describe, verbally and mathematically, your model for estimating the temperature in any given town. Do you think your model will always work? Are there factors that you didn't consider that professional meteorologists probably use in their own models?

ESTIMATING TEMPERATURES

Student Name: _____ Date: _____

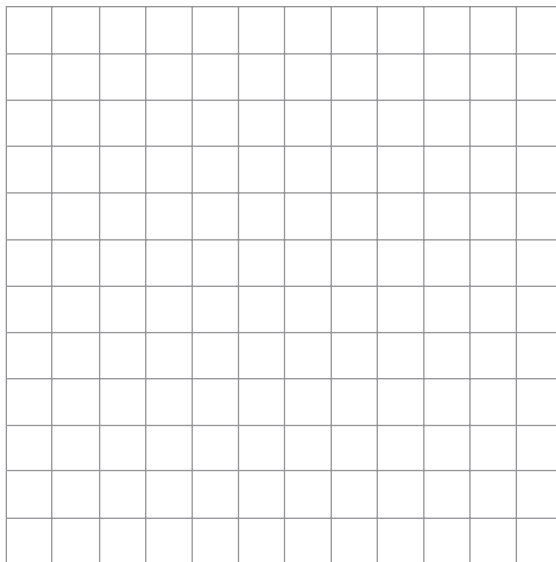
Using functions is one way to model the change in temperature. A **linear function** is a function that grows by equal distances over equal intervals. The amount that they change is called the *slope* and is usually denoted by m .

7. Is there a way that you can use a graph to represent the temperatures in the towns shown in question 1? Plot the towns as points on the coordinate plane. Draw a line containing the points.



*What should the values on the x-axis and y-axis represent?
What does the line between the points describe?*

8. Use the method above to graph and represent the situation from question 2. What are the coordinates of the middle point? Does this coordinate have any relationship with the temperature you estimated?

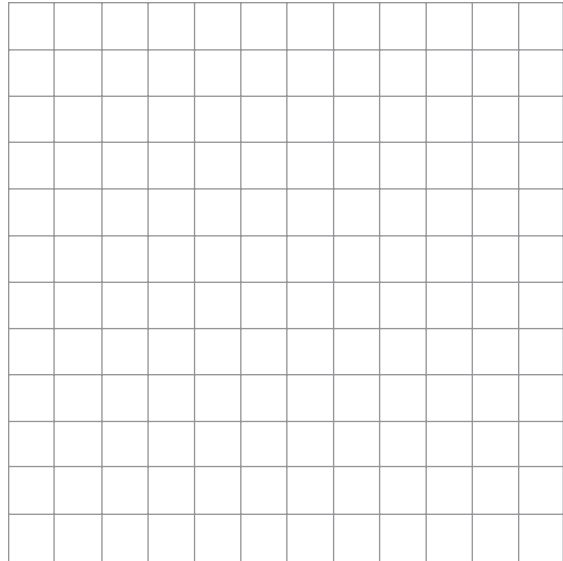
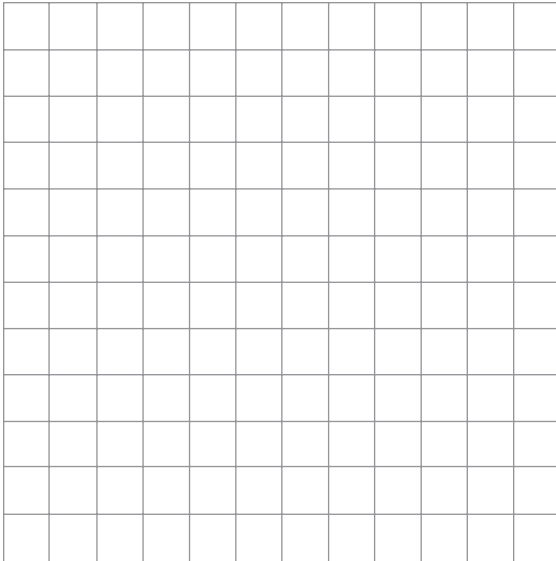


ESTIMATING TEMPERATURES

Student Name: _____ Date: _____

9. Modify your method for modeling the situation in question 4 by using graphs of linear functions. Does this model give the same result as in question 4? How is the rate of change in the temperature between two points described on the graph?

You may need to use more than one graph.



10. Describe how you used graphs of linear functions to model estimating temperatures at given points. What are the similarities and differences between your original model and the linear function model?

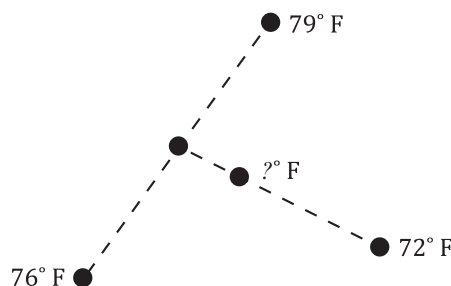
11. Use the work you've done to describe what is meant by "rate of change". How does it relate to the graph of a linear function? Is there a way to calculate or estimate the rate of change of a linear function easily?

ESTIMATING TEMPERATURES

Teacher's Guide — Possible Solutions

The solutions shown represent only some possible solution methods. Please evaluate students' solution methods on the basis of mathematical validity.

1. The temperatures here would seem to indicate that temperature changes continuously and constantly over intervals of equal length. That is, temperature appears to change linearly.
2. The only variable affecting the temperature, as far as we can see, is distance. Many other variables affect temperature, but those data are not given here. The student may be able to refine the model to include those variables later, if necessary. The temperature of the unknown is approximately 49°F .
3. The temperature changes at the same rate over equal distances. Let a , b , and x represent the temperature in degrees at points A , B , and X , respectively. If an unknown temperature point, X , lies between (collinearly with) two known temperature points, A and B where A is the lower temperature, then $x = (AX/AB)(b - a) + a$.
4. The model found in question 3 can be used twice. First, construct a line between any of the two known points (the line between 79° and 76° is shown). Second, construct a line through the last known point and the unknown point. Use the model to estimate the temperature at the point of intersection of the two lines. Finally, use that estimation to estimate the temperature at the desired point.
5. The model from question 4 can be used with any 3 known points. Students should find that different sets of known points produce different answers. They may conclude that the set of closest known points should be used or that the average of the answers for all sets of three known points should be used.
6. A model description is given in the solution to question 4. The topography of the area is one major variable that has been left out of the model. Hills and valleys affect the flow of air and, hence, temperatures.
7. The answers to the previous questions are replicated in the context of a linear graph.
8. The answers to the previous questions are replicated in the context of a linear graph.
9. The answers to the previous questions are replicated in the context of a linear graph.
10. Linear functions describe the rate of change (in their slope) of the temperature. The distance is the x -value and the temperature is the y -value.
11. For a linear function, the **slope** is the rate of change.



ESTIMATING TEMPERATURES

Teacher's Guide — Extending the Model

Suppose you wanted to estimate a temperature outside the intervals in questions 1 and 2. What would you do? Try an example in which the numbers are not just monotone, but with the perturbation of some “noise”. (What might cause such “noise”? Changes in elevation, wind, etc.) The basic pattern still looks linear. Fit a line to the data as well as you can. This idea can lead to the method of least squares.

No two of the three given temperature stations have the same temperature. Therefore, one of the three numbers must be between the other two. In this case, 76°F is between 72°F and 79°F . Where on the line between 72°F and 79°F is the temperature also 76°F ? Find that point and connect it by a straight line to the vertex with temperature 76°F . All lines of constant temperature will be parallel to this one. Fill in these lines for all whole-number temperatures between 72°F and 79°F . Now estimate the temperature at the point marked $S^{\circ}\text{F}$. If two of the original three temperatures were the same, how would you modify the procedure you just found? What is now the direction of lines of constant temperature?

If the point marked $S^{\circ}\text{F}$ were outside the triangle, how would you estimate its temperature? Draw the points with temperature 72°F , 76°F , and 79°F on a flat surface, and construct a vertical post of heights 2, 6, and 9 (ignoring the 7) at each of these points. Lay a flat surface on top of these three posts. How does the height of the point marked $S^{\circ}\text{F}$ compare with the height you estimated before? Draw lines of constant height onto your surface. How do they compare with the lines you drew before?

You now have four points whose temperature are known. Take any three of these points and use them to estimate the temperature at S as you did above. Use a different set of three points and do it again. How many such sets of three points are there? Look at the guesses for the temperature at S that you now have. Are they equal? If not, order them. Can you convince your parents that the temperature will be between the highest and the lowest of these? How do you feel about the average of the four?

BENDING STEEL

Teacher's Guide — Getting Started

Nicholas H. Wasserman
Southern Methodist University

Purpose

Metal railroad tracks expand and contract due to weather. In this two-day lesson, using the assumption that a railroad track is secured at both ends, students will use models to estimate how expansion of the track affects the height of the rail off the ground. Sometimes tracks will expand outward along the ground, but this lesson focuses on the case where they expand upward.

Interestingly, very small increases in length as a result of expansion have a large effect on height. Students will investigate this phenomenon using both triangular and arc models.

Prerequisites

Students should know conversion of units, systems of equations, properties of circles, and basic trigonometry.

Materials

Required: Graphing calculators.

Suggested: None.

Optional: Any materials to build physical models (e.g., clay, ice pop sticks, cardboard, paper, plastic rulers, etc.).

Worksheet 1 Guide

The first three pages of the lesson constitute the first day's work. The situation is explained to the students and they work at creating a simple model to describe the track length and height upon expansion. Students estimate how temperature increases affect the total length of the railroad track. (Students should be aware of the units – both feet and meters – and the conversions between them.) Students use this information to create an initial model to determine how high the tracks would rise off the ground. Students often choose to model the track expansion with an isosceles triangle; this model will be refined on the second day.

Worksheet 2 Guide

The fourth and fifth pages of the lesson constitute the second day's work. Students are challenged to determine if their model "overlooks" too much information. A railroad track would have to bend or curve when its length is expanded, so an isosceles triangular model, for example, will not suffice. Students refine their model to use an arc (of a circle) to model the track. Students use properties of circles, arc length, and basic trigonometry to design a system of two equations. Students should describe the original length (a chord of a circle), which is known, in terms of the unknown radius and central angle using basic trigonometry. Students should also describe the arc length, which is known, in terms of the unknown radius and angle of the arc. This system of equations should allow students to solve for the two unknowns, the radius and central angle, and to determine the missing height.

CCSSM Addressed

F-IF.4: For a function that models a relationship between two quantities, interpret key features of graphs and tables in terms of the quantities, and sketch graphs showing key features given a verbal description of the relationship.

F-IF.6: Calculate and interpret the average rate of change of a function (presented symbolically or as a table) over a specified interval. Estimate the rate of change from a graph.

G-MG.1: Use geometric shapes, their measures, and their properties to describe objects.

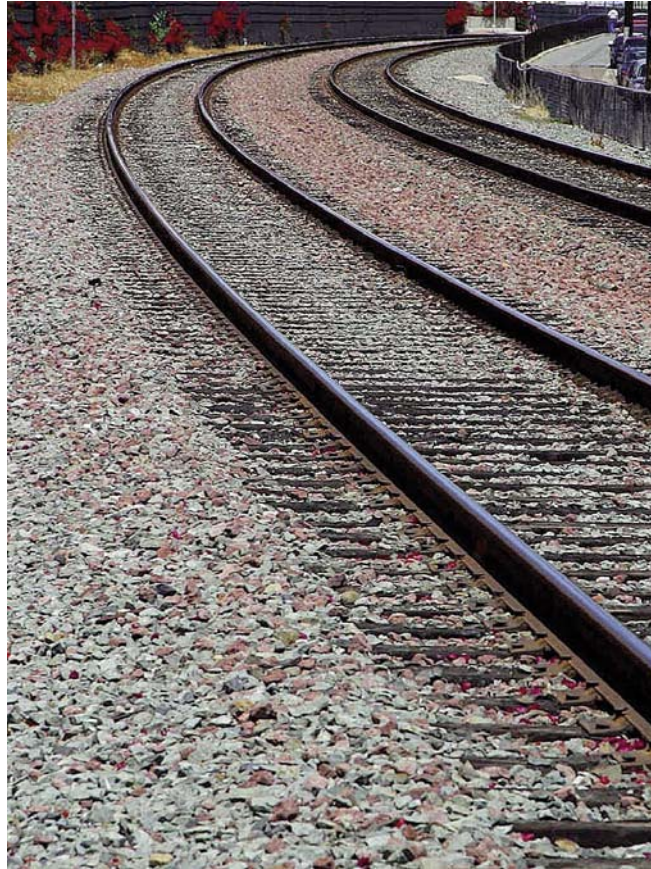
G-MG.3: Apply geometric methods to solve design problems.

BENDING STEEL

Student Name: _____ Date: _____

Railroads are a common source of transportation around the world. Because the tracks are made of metals (often steel), they expand and contract due to changes in temperature and various problems arise.

Suppose a section of track is fastened down at both ends. The natural process of heating and cooling causes the track to expand and contract. If the track length increases, but is nailed down at both ends, then the tracks should rise off the ground. The tracks may also expand outward along the ground, but this lesson focuses on the case where they expand upward.



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Leading Question

How can railroad designers design tracks that stay safely on the ground in all types of weather?

BENDING STEEL

Student Name: _____ Date: _____

1. The world's longest railroad sections are about 120 meters in length, or about 400 feet, with the typical length in the United States less than 100 feet. Suppose in your city that temperature changes on average about 45°F (25°C) from a cold, winter day, to a warm, summer day. If the track is 120 meters in the winter, the climbing temperature and heat during the summer causes the tracks to swell and increase in length. The linear expansion coefficient, α , for steel is approximately 0.000002 meters per degree change in temperature (°C). Use this information to determine how much the track expands in length between winter and summer. Convert your answer to feet and then to inches.

What is the meaning of "linear expansion coefficient"? What does it help you determine in relation to this problem?

2. Draw a model of how you think the 400 foot track would look if its length expanded by the amount you found in question 1. Label all the known lengths.

Does your model look like a familiar mathematical shape?

3. How high off the ground do you think the track would rise? Give an estimate and explain your thoughts.

BENDING STEEL

Student Name: _____ Date: _____

Did your model for railroad track expansion seem reasonable? Can you imagine railroad tracks rising as far off the ground as you determined? In mathematical modeling, one should always check to see if the proposed model is reasonable in the real world. If not, it often serves as a good “starting point” and as a good guide for a new, revised model — after all, one should always learn from mistakes!

7. Based on real-life, physical models, it seems reasonable to model track expansion as the arc of a circle. Draw an arced model below, labeling the original straight length (a chord), and the new curved length. Extend the arc to draw the circle that contains it. Label the unknown radius, r , and central angle, θ , of the circle.

8. Design of a system of two equations to help you determine r and θ . Solve for the two unknowns.

What two equations can you write that will help you solve for r and θ ? What do you know about them?

BENDING STEEL

Student Name: _____ Date: _____

9. Using the identified values for the radius, r , and central angle, θ , that are required for an arced model of this situation, how high off the ground would the tracks rise? Is the result surprising or what you expected?
10. Compare your first model and the arced model. How different are the results? Did either of the results surprise you? Did either result seem unreasonable? Which of the models do you think works better and why? Was the extra work required to make the arced model “worth it” considering the results found?
11. It seems that very small changes in length due to changes in temperature cause very large changes in height. Engineers have avoided this problem in railroad tracks, bridges, and other structures by doing something very simple. Can you find an easy solution to avoid railroad tracks being lifted several feet in the air due to expansion from the weather? What is it?

BENDING STEEL

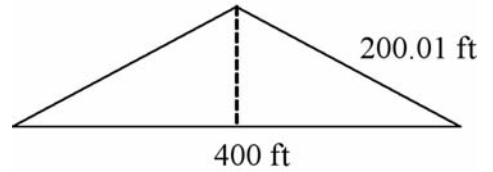
Teacher's Guide — Possible Solutions

The solutions shown represent only some possible solution methods. Please evaluate students' solution methods on the basis of mathematical validity.

1. The length increases $0.000002 \cdot 25 \cdot 120 = 0.006$ m, which is approximately 0.02 ft (roughly 0.25 in).

2. One reasonable initial model is an isosceles triangular model.

The length of the base remains the same while the length of each side is determined by half of the sum of the increase in length and the initial length (the length of the base). (Shown to the right, not to scale.)



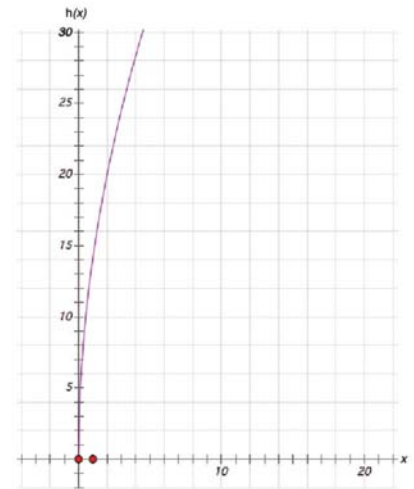
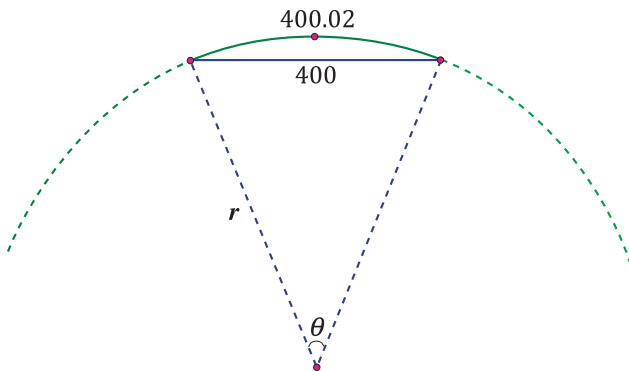
3. Answers will vary. Most students will expect the height to increase only slightly, probably less than 0.25 in.

4. $\sqrt{(200 + (0.5)(0.25/12))^2} - 200^2 = 2$ ft. A total increase in 0.25 inches in length results in a 2 foot increase in height at the middle, which is 96 times as large as the increase in length.

5. $h(x) = \sqrt{(200 + 0.5x)^2} - 200^2$. The sketch of the graph is shown to the right.

6. Given very small changes in x , near the origin, the height changes quickly. The slope of the curve is very steep near the origin.

7. The arced model is drawn below (not to scale).



8. The two equations in the system are $\frac{\theta}{360} \cdot 2\pi r = 400.02$ and $\sin\left(\frac{\theta}{2}\right) = \frac{200}{r}$.

These equations yield $\theta = \frac{360(400 + 0.25/12)}{2\pi r}$ and $\theta = 2\sin^{-1}\left(\frac{200}{r}\right)$. The result is that $r = 11,314.5$ ft and $\theta = 2.0256748^\circ$.

9. The height in the middle of the arc is $h = 11,314.5 - 11,314.5 \cdot \cos(1.0128374^\circ) = 1.768$ ft.

10. There is very little difference between the two models: approximately 3 inches. As expected, the arced model reduces the height, but not by much. An interesting discussion can revolve around the increased accuracy versus the extra time and effort expended between the two models.

11. Leave space between the railroad tracks: the use of expansion joints is ubiquitous in building and designing railroad tracks, bridges, and other structures because of this problem.

BENDING STEEL

Teacher's Guide — Extending the Model

Please note the process of refinement for the model in this problem. The phenomenon is familiar, the mathematical fact is that the answers with a triangular and a circular model are surprisingly large but also close to each other, and the physical fact is that rails are laid with expansion joints.

The linear expansion coefficient for steel is given as approximately 0.000002 meters per degree centigrade. The accuracy implied by that figure does not justify the number of decimal places in the solutions given to questions 8 and 9. On the other hand, this is a great opportunity to discuss the number of significant figures that *does* make sense, and the extra digits help to check that the right computation was entered into the calculator, even if the answers were copied to too many places.

There is a simpler and more domestic situation which leads to the same kind of mathematical phenomenon that underlies the problem solved by expansion joints in railroad tracks. But first, a message from our sponsor, namely, mathematics.

Here is the mathematical phenomenon: if you have a right triangle whose hypotenuse H is just a tiny bit longer than its longer leg L , then the length S of the shorter leg is incredibly sensitive to the accuracy of H , or, more precisely, the accuracy of the difference between H and L . Why? The formula for S is given by

$$S = \sqrt{H^2 - L^2}$$

whose partial derivative with respect to H is $\frac{\partial S}{\partial H} = \frac{H}{\sqrt{H^2 - L^2}}$.

We now note that $H^2 - L^2$ can be factored into $(H + L)(H - L)$. Then $\frac{H}{\sqrt{H+L}}$ is almost $\sqrt{\frac{H}{2}}$ as H

approaches L^+ . Hence $\frac{\partial S}{\partial H}$ is almost $\sqrt{\frac{H}{2(H-L)}}$ and we notice that it becomes arbitrarily large as H approaches L^+ .

This is why you see what you see in the plot accompanying the discussion of question 6, namely why S becomes large so rapidly as H gets a tiny bit larger than L . A small error in the abscissa can lead to a large error in the ordinate.

The domestic situation referred to above concerns the hanging of a small picture on a wall at the precise height which has been recommended by the spouse of the person doing the hanging. Typically, that person might screw two small eye screws into the two vertical sides of the back of the picture frame, run a taut string or wire between the screws, and then put a nail into the wall so that the bottom of the picture will be at the preordained height when the picture hangs from the nail at the middle of said string or wire. Because of the weight of the picture, the length of the string/wire will be a tiny bit greater than the distance between the screws, and even a very accurate measurement of that length will lead to a large error in S , and it is S that determines the height of the picture. The spouse, of course, may be disinclined to transfer the blame for the inaccurate height from the spouse to a partial derivative going to infinity.

A discussion of this problem, and another almost equally unstable version with a heavy picture hanging from a molding, can be found in COMAP's *Consortium*, Number 85, Fall/Winter 2003, pp. 3–4.

A BIT OF INFORMATION

Teacher's Guide — Getting Started

Nicholas H. Wasserman
Southern Methodist University

Purpose

In this two-day lesson, students will learn to use a logarithmic function to model information functions. A significant portion of the secondary curriculum revolves around the analysis of functional relationships. In the context of computers, the notion of sending and receiving information gives way to an interesting relationship between the required length of code and how much information it carries.

In fact, this represents one of very few real world situations where only a logarithmic function can model the relationship.

Prerequisites

Students should be familiar with functional and inverse relationships and know the properties of exponents.

Materials

Required: None.

Suggested: Calculators.

Optional: Candy or another manipulative (to identify a specific type out of several).

Worksheet 1 Guide

The first three pages of the lesson constitute the first day's work. Students begin the activity by becoming familiar with the notion of bits — how computers send and receive information. A simple question and answer game is used to demonstrate how many "questions" or "bits" of information it takes to identify one item out of many. While playing, students should be encouraged to devise a logical model for *finding* the correct item — not a way of *guessing* it. Students then identify three principles that govern an information function.

Worksheet 2 Guide

The fourth and fifth pages of the lesson constitute the second day's work. Based on the three principles of the information function, students investigate and build up specific answers to identify the one function that can model this relationship.

CCSSM Addressed

F-IF.4: For a function that models the relationship between two quantities, interpret key features of graphs and tables in terms of the quantities, and sketch graphs showing key features given a verbal description of the relationship.

F-BF.1: Write a function that describes a relationship between two quantities.

F-BF.5: (+) Understand the inverse relationship between exponents and logarithms and use this relationship to solve problems involving logarithms and exponents.

A BIT OF INFORMATION

Student Name: _____ Date: _____

Did you know that the on/off symbol on a computer is a combination of a 0 and 1?

In computer language — a world of 0s and 1s — the ability to communicate and understand information depends on a mathematical function. It is customary to use the term “bits” to describe information: the usage of “bit” to describe sending information is actually short for “**binary digit**,” i.e. 0 or 1. A bit is actually a *unit*: one bit is the smallest possible building block of computer data, meaning that it can be communicated by a single binary digit . . . a 0 or a 1.



Leading Question

To get a basic sense of communicating information, if a computer is trying to communicate one of the numbers 1–50 to another computer, how many “yes” or “no” questions would it take for the second computer to identify the number correctly?

A BIT OF INFORMATION

Student Name: _____ Date: _____

When thinking about how much information it takes to communicate something, one analogy is how many “yes” or “no” questions you would have to ask to identify one object. Each “yes” or “no” would correspond to a 1 or a 0, respectively. How many bits of information it takes is the number of questions you have to ask. The string of 0s and 1s describes the sequence of answers to the questions asked.

1. Work with another person. One person should pick a number between 1 and 20. The other should ask “yes” or “no” questions until they guess the number. For every question, record a “0” for “no” and a “1” for “yes”. Try this several times. Is there a logical way to find out the answer (without guessing at random!)? If so, what is it? What does the string of numbers represent?

What types of questions are more useful — guessing specific numbers or guessing a range of numbers?

2. Should the string of numbers in question 1 ever be longer than 20 digits? Explain why or why not. Explain what a string consisting of a single digit would represent. How long do you think the string of numbers should be on average?

A BIT OF INFORMATION

Student Name: _____ Date: _____

The number of bits is the same as the number of questions that **MUST** be asked in order to “whittle down” the correct answer in an efficient way. (No guessing at random!) This means that if you had to ask, for example, 9 questions, the answer would have “cost” you 9 bits. We will use n to represent how many items there are to choose from. In question 1, $n = 20$ since there were 20 possible numbers. A functional relationship exists between n and the number of bits required to communicate this information. Let $f(n)$ represent how many bits of information, i.e. how many 0s or 1s, are required to specify one item out of n possible items.

In questions 3 – 5, you will work out the properties of the function.

3. What is the numerical value of $f(2)$? $f(1)$? Explain your reasoning. Describe what $f(50)$ means.

4. Communicating ONE thing out of MANY possibilities takes a certain amount of information. If you are trying to communicate ONE thing out of MANY MORE possibilities, what is the effect on the amount of information? What does this mean regarding a property of the function?

5. One way to identify something is to start with the whole group and look for the answer. Another way is to *split* the large group into groups of 2 or 3 or m , and determine how much information it would take to identify which of these groups the thing is in, and then figure out how much information it would take to specify the ONE thing from within that group (however big it is . . . say n). Using the second method, what does $f(m \cdot n)$ equal?

A BIT OF INFORMATION

Student Name: _____ Date: _____

6. Using aspects of the three properties you found in questions 3 – 5, identify values for $f(4)$, $f(8)$, and $f(16)$. Considering your answer for question 5, what is $f(m^k)$?

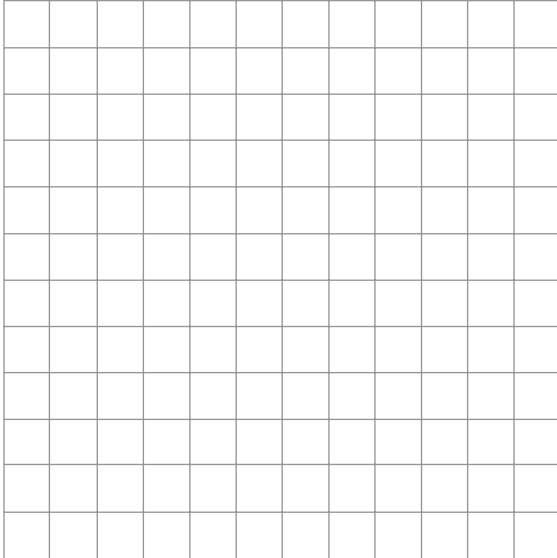
7. Estimate $f(3)$ and $f(5)$ based on what you already know.

8. How precise are these estimates? In particular, consider how averaging the number of questions it might take to guess an object out of 3, or bundling a few sets of 3 objects into one, might affect the numerical possibilities for $f(3)$. Likely, decimal values would make sense regarding the value of $f(3)$. So, how precise can you be? Since $3^2 > 2^3$, then $f(3^2) > f(2^3)$, and so $2f(3) > 3f(2)$. This means that $f(3) > 1.5$. Using a similar process, try to get a better estimate for $f(3)$ and $f(5)$ than you did in question 7. How close can you get?

A BIT OF INFORMATION

Student Name: _____ Date: _____

9. What does the graph of this function look like? Sketch it below. Have you seen a graph that looks similar to this one before? Do you recognize a function with these properties?



Finally, the minimum number of bits of information it takes to identify one item from n objects, is

$$f(n) = \underline{\hspace{10em}} .$$

A BIT OF INFORMATION

Teacher's Guide — Possible Solutions

The solutions shown represent only some possible solution methods. Please evaluate students' solution methods on the basis of mathematical validity.

1. Answers will vary. An efficient method is to split the set of possibilities into equally-sized subsets and ask if the correct solution is in one of the subsets. Repeating this process will, for 20 possible items, usually yield about 4 or 5 questions. No string should be shorter than 1 or longer than 20. The string of numbers represents both the number of questions asked and the sequence of answers.
2. No, the string should not be longer than 20 digits; there were only 20 possible correct solutions. A single digit represents a correct initial guess. The average length will be around 4 or 5.
3. $f(2) = 1$ because for two items, it should only take 1 question, a "0" or a "1" to designate between the two possibilities. $f(1) = 0$, since it should not take any information to guess one item. $f(50)$ would be the number of "bits" required to identify one item out of 50.
4. More possibilities mean more information is necessary. The function is strictly increasing.
5. $f(m \cdot n) = f(m) + f(n)$. If you split the original number of objects into m -sized groups, then it should take you $f(n)$ bits to figure out which group, and then $f(m)$ bits to identify the single item within that group.
6. $f(4) = 2$; $f(8) = 3$; $f(16) = 4$; $f(m^k) = kf(m)$
7. $1 < f(3) < 2$; $2 < f(5) < 3$.
8. Similarly, you might use that: $2^5 > 3^3$, $2^6 < 3^4$, $2^8 > 3^5$, etc.; or $2^5 > 5^2$, $2^7 < 5^3$, etc.
9. The graph of the function is logarithmic. Some students may recognize its relation to the graph of an exponential function. $f(n) = \log_2(n)$.

A BIT OF INFORMATION

Teacher's Guide — Extending the Model

This model has been about measuring the information gained when you find out which one of n equally likely possibilities is the correct one. The function $f(n)$ which expresses the information gain was first developed by Claude Shannon in the late 1940s. We have seen that $f(n)$ must have the properties that

- (i): $f(1) = 0$;
- (ii): $f(n)$ is a monotone increasing function of n ;
- (iii): $f(m \cdot n) = f(m) + f(n)$.

If another property, (iv): $f(2)$ is defined to be 1 bit, then the function $f(n) = \log_2(n)$ is the only function with these properties. Thus, for example, $f(4)$ must be 2, $f(2) = 1$, and therefore $f(3)$ must be strictly between 1 and 2. We have estimated $f(3)$ as a little more than 1.5.

What does such a non-integer function value mean? We can give an estimate of $f(3)$ by thinking as follows: There are three equally likely possible outcomes a , b , and c . Suppose you ask: Is it a ? If that's correct, which has probability $1/3$, you have found it in one question; if that's not correct, which has probability $2/3$, it will take you one more question to tell whether it is b or c , so in that case it will have taken you two questions. So the expected number of questions is $(1/3) \times 1 + (2/3) \times 2 = 5/3$, which is approximately 1.67.

That's only an upper bound for $f(3)$, but our value $f(4) = 2$ is exact. Why the difference? Because when we take four equally likely possibilities a , b , c , and d , and divide them into two groups of two, these groups are again equally likely. But when we divide three possibilities a , b , and c into a versus the set consisting of b and c , these two groups are not equally likely, and the partition is inefficient. It's just like the game of twenty questions: The fastest way towards the answer is to ask questions whose answer is as nearly equally likely to be "yes" and "no" as you can make it.

You can get a better estimate for $f(3)$ by imagining that you have *two* batches of three and want to find the correct choice in each one. If you do them separately as two threes, it will take an expected number of $2(1.67) = 3.33$ questions. But you can do it in fewer questions: If the choices in the first batch a , b , and c and in the second batch d , e , and f , make a single batch of nine choices ad , ae , af , bd , be , bf , cd , ce , and cf . You can then divide the nine into batches of four and five — which have the advantage that they are more nearly equal than batches of size 1 and 2. The answer to the batch of four will be "yes" with probability $4/9$ and "no" with probability $5/9$. Divide the batch of four into two batches of two and the batch of five into batches of size two and three, and you will get that the expected number of questions is $29/9 = 3.22$, which is a definite improvement over (that is, "under") 3.33. By aggregating more and more problems into one, you can come closer and closer to $f(n)$.

You are now better prepared (we hope) for the derivation of **Shannon's formula**.

Here is a proof that a function $f(n)$ which satisfies (i) – (iv) above must be $\log_2(n)$. If n is a power of 2, then the answer follows from (iii) and (iv). This is the case where exactly equal division is possible all the way to the answer. So assume that n is NOT a power of 2. Now take an arbitrary power k of n — think of k as large. It must be that n^k is strictly between two consecutive powers of 2, say s and $s + 1$. In symbols,

$$2^s < n^k < 2^{s+1}.$$

Then

$$\begin{aligned} f(2^s) &< f(n^k) < f(2^{s+1}) \\ s \cdot \log_2(2) &< k \cdot f(n) < (s+1) \cdot \log_2(2) \\ \frac{s}{k} &< f(n) < \frac{s+1}{k}. \end{aligned}$$

A BIT OF INFORMATION

Teacher's Guide — Extending the Model

Now do the same with n^k itself. We get

$$2^s < n^k < 2^{s+1}$$
$$s \cdot \log_2(2) < k \cdot \log_2(n) < (s+1) \cdot \log_2(2)$$

$$\frac{s}{k} < \log_2(n) < \frac{s+1}{k}.$$

So the two quantities $f(n)$ and $\log_2(n)$ are between the same two bounds, and these bounds differ by only $1/k$. Hence

$$|f(n) - \log_2(n)| < \frac{1}{k},$$

where, as you remember, k is arbitrary. The only way to satisfy this is to have $f(n) = \log_2(n)$.

Given the earlier example, you can see why the proof works. We have batches of n , and we put k such batches together so that there are now n^k possible outcomes. The “cost” in questions “*per batch*” is now arbitrarily close to $f(n)$.

The mathematics of information theory begins with these ideas. They extend, for example, to outcomes which are not assumed to be equally likely and to situations where the possibility of errors “contaminates” the responses.

STATE APPORTIONMENT

Teacher's Guide — Getting Started

Andrew Sanfratello
Mohegan Lake, NY

Purpose

A new country is being formed in this two-day lesson. Students will determine how to allot the representation for the different states in the country, also known as apportionment.

Begin by asking students how democracy works in the US. Ask them how a country that is newly forming and wishes to adopt a similar representation system to the US might pick how many representatives each state gets. What different *mathematical* ways are there to model this?

Prerequisites

Students should be familiar with percentages and ratios.

Materials

Required: Internet access for searches on the Hamilton and Jefferson Methods.

Suggested: Spreadsheet software (such as Excel).

Optional: None.

Worksheet 1 Guide

The first three pages of the lesson constitute the first day's work. Students are introduced to a fictional country with four states and asked to determine how the states should apportion their representation. They should be encouraged to try different methods, and then on the third sheet of the day, they are asked to investigate the Hamilton and Jefferson Methods that were introduced early in United States' history as a way to allot representatives among the states.

Worksheet 2 Guide

The fourth and fifth pages of the lesson constitute the second day's work. Here, students are introduced to the Quota Rule, and then asked to interpret their model from the prior day with the new rules in place. Additionally, a small change is made to the population distribution that changes which of the models might be more efficient. Finally, students are urged to use the current US system of state apportionment, create a recursive function from this (here is where spreadsheets can be used), and determine the pros and cons of each of the state apportionment methods.

CCSSM Addressed

F-BF.1: Write a function that describes a relationship between two quantities.

STATE APPORTIONMENT

Student Name: _____ Date: _____

In the United States House of Representatives, the number of seats that each state receives is based on the population of the state. Each state is guaranteed at least one representative, but after that it is determined solely by the number of people living in the state according to the census taken every ten years. There have been many different ways that the US state apportionment has been determined in the past.



Source: www.2010.census.gov

Leading Question

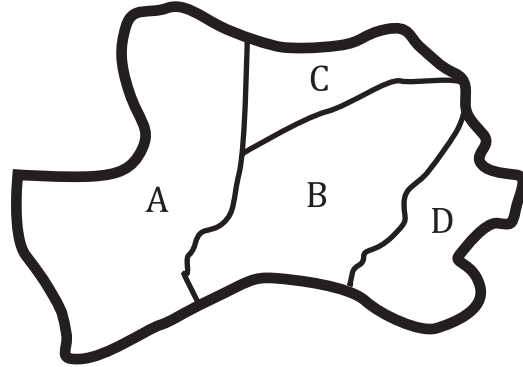
How might you arrange a system so that each state is represented fairly? What obstacles do you think might be present?

STATE APPORTIONMENT

Student Name: _____ Date: _____

For simplicity, imagine that a newly formed country wishes to copy the US House of Representatives. This new country has just 100,000 people split up into only four different states, listed in the table below.

State	Population
A	15,000
B	17,000
C	28,000
D	40,000



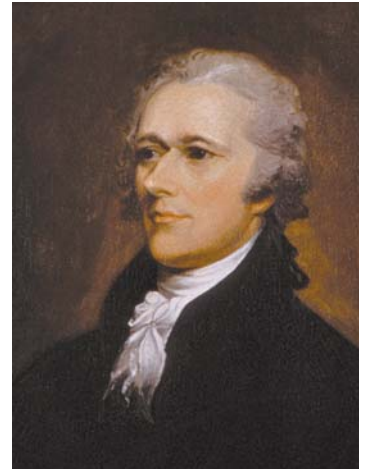
1. If the new country plans on having 25 representatives in its House of Representatives, how many should each state receive? What if they plan to have only 17 representatives?
2. How did you calculate how many representatives each state should receive? Did you use the same method for both 25 and 17 representatives?
3. Can you create a method that is fair to all states in both cases? Describe how your method works and why you believe it to be fair.

STATE APPORTIONMENT

Student Name: _____ Date: _____

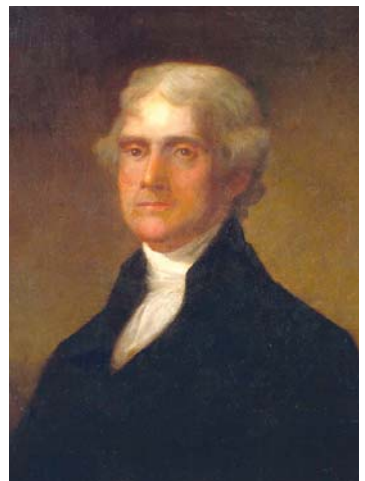
4. Which states (if any) would disagree with the apportionment that you have created in each of these cases? Do both scenarios create the same problems?

5. The **Hamilton Method** was devised by Alexander Hamilton as a technique for fair apportionment. Investigate what the Hamilton Method was and if you agree or disagree with its fairness. Do either of your methods share any similarities with the Hamilton Method?



Public Domain:
John Trumbull (1756–1843)

6. Thomas Jefferson also devised his own method at the same time that Alexander Hamilton did. Research the **Jefferson Method**. What are the differences and similarities between the Jefferson Method and the Hamilton Method? Does the Jefferson Method compare with either of your methods? Which of the two methods is better suited for this model?



Gwillhickers (Whitehouse portrait gallery)
[Public domain], via Wikimedia Commons

STATE APPORTIONMENT

Student Name: _____ Date: _____

The **Standard Quota** is a number assigned to each state that is calculated by taking the percentage of the country's population that live in that state, multiplied by the nation's number of representatives.

The **Quota Rule** says that each state will receive one of the whole numbers that the Standard Quota falls between as their number of representatives. If the standard quota is a whole number, then the number of representatives must be the same as the Standard Quota.

7. Do your methods from the previous day follow the Quota Rule? If not, might you be able to alter them so that they do?

8. Suppose that 1000 people move from state B to state A. How would this affect your earlier models with both 25 and 17 representatives? Which one is better suited now? Is it the same as before the movement? Should a reasonable model have to change so dramatically when a small number of people move, as is the case in this example?

State	Population
A	16,000
B	16,000
C	28,000
D	40,000

STATE APPORTIONMENT

Student Name: _____ Date: _____

9. The current method used by the US uses the **geometric mean** as the denominator and the state's population in the numerator in a recursive formula. Go back and use the method that the United States uses in their apportionment for the new country with the original and new populations. Does this method work well?

10. What might be some other methods to determine fair apportionment? What problems, if any, arise with other methods? Which of the apportionment methods do you think is fairest? The US House of Representatives has 435 representatives. Does this make sense? There are a number of paradoxes that exist with state apportionment. What are they and which ones arise with which models?

STATE APPORTIONMENT

Teacher's Guide — Possible Solutions

The solutions shown represent only some possible solution methods. Please evaluate students' solution methods on the basis of mathematical validity.

1. With 25 representatives, the states should receive the following apportionment: A = 4; B = 4; C = 7; D = 10. With 17 representatives, the states should receive the following apportionment: A = 2; B = 3; C = 5; D = 7.
2. Answers will vary, but you should focus on what to do with any fractional values left over.
3. Answers will vary. In general, methods can be created that are mostly fair but some unfairness remains.
4. State B would likely disagree with the apportionment with 25 representatives, as they are receiving the same representation as state A even though they have a larger population.
5. The Hamilton Method always gives the states with the highest fractional Standard Quota the extra seat(s).
6. The Jefferson Method involves modifying the *divisor*, d , which is calculated by taking the quotient of the total population and the number of seats. d is then decreased until the quotient of each state's population and the new d add up to the exact number of seats needed.
7. The Jefferson Method violates the Quota Rule at times. The Hamilton Method does not.
8. Under the new population, with 25 representatives, the states receive the following apportionments: A = 4; B = 4; C = 7; D = 10. With 17 representatives, the states should receive the following apportionments: A = 2; B = 3; C = 5; D = 7. In the latter apportionment, states A and B have the same population, but do not receive equal representation.
9. Using the geometric mean eliminates both of the issues that came up in questions 1 and 8.
10. Answers will vary. The various paradoxes are known as the Alabama paradox, the population paradox, and the new-state paradox. Two additional methods have also been used or proposed in US history. They are the Webster Method and the Adams Method. More information about state apportionment, can be found on the websites listed below:

<http://www.cut-the-knot.org/ctk/Democracy.shtml>

http://www.census.gov/history/www/programs/demographic/methods_of_apportionment.html

<http://www.census.gov/population/apportionment/about/index.html>

<http://www.ctl.ua.edu/math103/apportionment/appmeth.htm>

STATE APPORTIONMENT

Teacher's Guide — Extending the Model

The terminology and notation for apportionment are taken from the problem of determining how many seats in the House of Representatives each state should receive. Other applications arise frequently: how to determine how many teaching slots — or computers — each department in the high school should get, or how to divide the US Navy among several oceans! The notation is usually taken from apportioning the House of Representatives:

Let s be the number of states, and let p_1, p_2, \dots, p_s be the populations of the states. Let a_1, a_2, \dots, a_s be the number of seats each state receives, and let h be the size of the house.

Thus, $\sum_{i=1}^s a_i = h$. The **exact quota** for state i is $q_i = p_i h \div \left(\sum_{j=1}^s p_j \right)$.

Unfortunately, this number is almost never an integer, and so we define the **lower quota**, denoted by $\lfloor q_i \rfloor$, as the integer part of q_i and the next integer above or equal to q_i as the **upper quota**, denoted by $\lceil q_i \rceil$.

The problem of apportioning is to determine what the functions f_j will be that take the population vector \vec{p} and the house size h and produce the apportionments a_j . In other words, we want $a_j = f_j(\vec{p}, h)$. The modeling arises with a vengeance when you begin to ask precisely what properties you would like the functions f_j to have. For example, you would probably like four properties:

Property 1: a_j is always between the lower quota and the upper quota for state j . We would say that the apportionment method “satisfies quota”.

Property 2: If the house size h is increased and nothing else changes, then no state's number of seats should decrease. We would say that the apportionment method is “house monotone”.

Property 3: If the population of state i does not decrease relative to that of state j , then it should not happen that a_i decreases while a_j increases. We would say that the apportionment method is “population monotone”.

Property 4: If a new state is added, and the house size is increased by the number of seats for that state, then no other state's number of seats should change.

Other properties may of course be considered — and have been (believe me). It's an extensive and thrilling area of mathematical modeling. One of the astounding theorems which drives this subject, due to Balinski and Young, is the following — and it is one of the triumphs of mathematical modeling.

Theorem: There exists no method of apportionment which can guarantee both Property 1 and Property 3.

That's right! There is no way of being sure that our apportionment method satisfies quota and is population monotone. The best you can do, for example, is to try to minimize the probability of violating quota (whatever that means) under the condition that the method should be population monotone.

One of the discoveries students will make if they get into the subject of apportionment is that the arithmetic process of dividing is really a tricky business, and full of surprises.

RATING SYSTEMS

Andrew Sanfratello
Mohegan Lake, NY

Teacher's Guide — Getting Started

Purpose

In this two-day lesson, students will model rating systems like those used in many sports. They are asked to consider the various factors that the human mind employs to “rate” one team over another; they will then model a way to consider these factors in order to make a systematic, mathematical rating method. Note that even professional rating systems often are disputed for their “accuracy”: such is the nature of both mathematical modeling and sports!

Begin with the description of the situation: you are trying to compare teams or players, but not every team/player plays the other, so there is no clear “clean-cut” method. How can you devise a system to do this?

Prerequisites

Students should understand basic probability concepts such as the computation and meaning of “rate of success”. Students should be able to interpret the meaning of expressions in an equation or function.

Materials

Required: Internet access (for research), calculators.

Suggested: None.

Optional: None.

Worksheet 1 Guide

The first three pages of the lesson constitute the first day's work. Students consider the factors that they think should be included when comparing one team or player to another. They use this intuition to create a simple model for a rating system. Students are introduced to the Elo Rating System, one of the first systems of its kind, which was developed for chess players. They perform Internet research to determine what is included in the system and compare the system to their model, which they try to refine.

Worksheet 2 Guide

The fourth and fifth pages of the lesson constitute the second day's work. Students consider the different factors of the Elo system and make judgments about them based on both mathematics and intuition. Students then consider another rating system, RPI, and make decisions about its effectiveness based on their experience and intuition.

CCSSM Addressed

F-BF.1: Write a function that describes a relationship between two quantities.

S-MD.5: (+) Weigh the possible outcomes of a decision by assigning probabilities to payoff values and finding expected values.

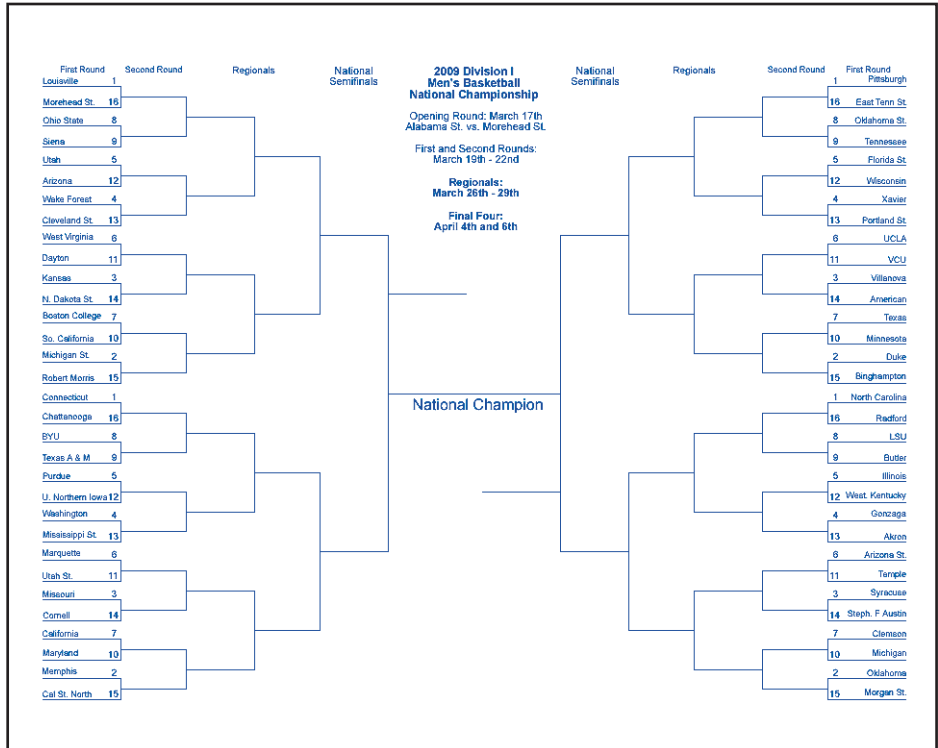
S-MD.6: (+) Use probability to make fair decisions.

S-MD.7: (+) Analyze decisions and strategies using probability concepts.

RATING SYSTEMS

Student Name: _____ Date: _____

In many professional games and sports, players or teams are rated in relation to others. This rating helps determine which players or teams are a good match for one another and helps determine who might win in a matchup between any two.



Leading Question

How would you devise a system to determine rating?

RATING SYSTEMS

Student Name: _____ Date: _____

Recall what your last model looked like and the information you found on the Elo rating.

7. How does the K-factor affect the rating of a player? Is it reasonable to use a fixed number? Did you use a fixed number in your initial rating system from question 3?

8. How is the Expected Result calculated? Did you use a similar mathematical method in your model?

9. The “Rule of 400” states that if two players are more than 400 points apart, then to determine the Expected Result, you assume that they are exactly 400 points apart. Why would this rule come to exist?

Think of the case where a very skilled player plays a bad, inexperienced player.

RATING SYSTEMS

Student Name: _____ Date: _____

10. In college basketball, RPI (Rating Percentage Index) is calculated with three factors: Winning Percentage (WP), Opponents' Average Winning Percentage (OWP), and Opponents' Opponents' Average Winning Percentage (OOWP). The weights used are 25%, 50%, and 25%, respectively. What similarities or differences does this have with the prior models?

11. A team has two options: they can play 5 other teams with an average winning percentage of 80% and an OWP of 90% and they'll likely win 1 out of the 5 games, or they can play 5 teams with an average winning percentage of 40% and an OWP of 50% and win 4 out of the 5 games. Which scenario will generate a greater RPI?

12. Is the weighting applied to the three factors in RPI appropriate? How might you change the weighting and/or include factors to alter the weighting?

RATING SYSTEMS

Teacher's Guide — Possible Solutions

The solutions shown represent only some possible solution methods. Please evaluate students' solution methods on the basis of mathematical validity.

1. Answers will vary but may include opponents' prior record and/or rating, the number of games played, "strength of schedule", and (if applicable) home/away records.
2. Generally, if Team A is rated and Team B is new and unrated, then if B beats A, B will have a higher rating than A. If A beats B, B will have a lower rating than A.
3. Answers will vary, but one way is to award the team or player with 1 point for a win, 0.5 points for a tie, and 0 points for a loss. The team or player with the most points will have the best rating.
4. This model does not account for unrated opponents. However, unrated teams or players may become rated by earning enough points to surpass a rated team or player.
5. K-factor is a number applied to a player (which varies according to the player's rating) and is used to balance highly rated players from increasing their rating easily. Expected Result is the expected score given a player's rating and the opponent's rating before they have actually played. Score is the actual result that occurs after the players have played each other. Scores in chess are 1 point for a win, 0.5 points for a tie, or 0 points for a loss.
6. The model above does not take into account various factors that the Elo system does. For example, it does not take into account opponents' rating or ability nor does it account for the advantage a highly rated player has over a very lowly rated or unrated player. It does, like Elo, consider ties to be "half-win" and "half-loss".
7. K-factor diminishes the value of individual games played by players with more experience. This causes more fluctuations with novice players' ratings, but more stability with expert players' ratings.
8. Expected Result is calculated for Player A with rating X against Player B with rating Y with the formula

$$E_A = \frac{1}{1 + 10^{\frac{Y-X}{400}}} .$$

9. The Rule of 400 prevents highly rated players from gaining points on their rating from playing people who are greatly below their skill level, and thus falsely boosting their rating.
10. Answers will vary, although as compared with the Elo system, both have three variables taken into account, although they are all quite different.
11. The first choice of games gives the team an RPI of 0.675 while the second choice of games (despite winning more of them) produces an RPI of only 0.525.
12. The RPI has a very high focus on "strength of schedule" — how well opponents perform — and performance of the team itself only accounts for 25% of the rating. Other factors to include might be based on score differences or how well a team does in "away" games, which are said to be more difficult to win.

RATING SYSTEMS

Teacher's Guide — Extending the Model

One of the outcomes of this lesson is a formula that is used for rating chess players: we have seen that the Expected Result for Player A when Player A with rating X plays Player B with rating Y is given by the formula

$$E_A = \frac{1}{1 + 10^{\frac{Y-X}{400}}}.$$

We see that it is based on a logistic curve. What do we expect from a formula like that?

- 1) If A is a lot better than B, we expect that it will give an answer close to 1. Well, suppose that X is a lot bigger than Y , say $X - Y = 360$. Then the exponent $(Y - X)/400$ is -0.9 , $10^{-0.9}$ is about $1/8$, so E_A is about 0.89. On the other hand, if we reverse the abilities of A and B, that is, set $X - Y = -360$, then $E_A = 0.11$. This fits our expectation of symmetry.
- 2) If the players are equal in ability, then $X = Y$, the exponent is 0, and $E_A = 0.5$. This fits our expectation of equality between the two players.
- 3) Suppose it has been observed that A wins about 1 time out of 3 against B. The difference in rating is expected to be $Y - X = 120$ since then the exponent is about 0.3, and $10^{0.3} = 2$. This expectation should work both ways: players that win 1 time out of 3 against an opponent (or equally rated opponents) should have a rating 120 less than these opponents, and also, players whose rating is 120 less than an opponent's should win 1 time out of 3.
- 4) This Expected Result can actually be interpreted as the probability that A will win plus half the probability of a draw. This implies that for all X and Y , $E_A + E_B = 1$. The algebraic exercise to show that this is true is not quite a one-step trivial exercise, and should not be missed.
- 5) All of this depends, of course, on how a player's rating is computed and adjusted. An important issue is human behavior given the rating system and the natural desire to improve one's rating.

How does a formula like this compare to a formula in physics, such as the formula for the range of a batted ball hit with velocity V at an angle a with the horizontal? Yes, that formula usually ignores air resistance and the height of the batter, but we can defend it on the basis of the principles of mechanics. We could correct for the height of the batter, and even for air resistance, and really believe the answer. The philosophy behind our formula for rating chess players is different. We want the formula to *act* as a probability, and to *behave* in certain limiting ways. We want it to agree with our rating system. If it has the effects we desire, we accept it, not because in some deeper sense we know it to be correct, but because it has the right shape and gives results we like and can use.

Mathematical models in many aspects of social science often satisfy similar expectations. The shape of the curve is right, the way we use it to optimize behavior or expenditures makes sense, and we don't expect the numbers to be exactly right. For example, a manufacturer expects that there is flexibility in the use of capital versus the use of labor in a given production program. If we have more machinery and automation, we have fewer laborers. People like to use a formula of the form

$$K^\alpha L^{1-\alpha} = C$$

where K stands for capital, L stands for labor, C is a constant, and α is an exponent between 0 and 1 chosen to be reasonable for the particular industry under consideration. We don't expect this formula to be exact, but it's the right shape and it fits real data at a couple of points pretty well. What can you do with it? Well, for example, you can draw a line of fixed expenditure on the same plot and get a pretty good idea of the mix of capital and labor that will give you the most product for your money. But don't forget maintenance!

This, as people like to say, is not rocket science, but it's typical of the kind of models that can be created and used outside of the "hard" sciences.

THE WHE TO PLAY

Shereen Khan & Fayad Ali
Trinidad and Tobago

Teacher's Guide — Getting Started

Purpose

In this two-day lesson, students develop different strategies to play a game in order to win. In particular, they will develop a mathematical formula to calculate potential profits at strategic points in the game and revise strategies based on their predictions.

Allow students to imagine that they are living in the twin islands of Trinidad and Tobago where a popular game called **Play Whe** is played everyday. They can think of the game as an investment opportunity and their goal is always to realize a profit. How can they devise a strategy so that their expenditure is always less than their potential winnings?

Prerequisites

Students should understand how to interpret graphs of linear and quadratic functions, how to generate number sequences, how to calculate simple probabilities, and have basic algebraic skills such as substitution and manipulation of symbols.

Materials

Required: Graph paper and scientific calculators.

Suggested: Software for generating tables and graphs.

Optional: Graphing calculators.

Worksheet 1 Guide

The first three pages of the lesson constitute the first day's work where students are first introduced to the traditional game from Trinidad and Tobago. Students analyze the gameplay and create a model to describe its profit. Then they revise that model to try to maximize their profit.

Worksheet 2 Guide

The fourth and fifth pages of the lesson constitute the second day's work and introduce students to arithmetic progressions in order to think of the problem algebraically. Upon examining their method from the first day, they are asked to observe what happens when different variables are altered in the formula. Students ultimately are led to question whether an ideal method of betting is possible.

CCSSM Addressed

F-BF.2: Write arithmetic and geometric sequences both recursively and with an explicit formula, use them to model situations, and translate between the two forms.

F-LE.2: Construct linear and exponential functions, including arithmetic and geometric sequences, given a graph, a description of a relationship, or two input-output pairs (include reading these from a table).

THE WHE TO PLAY

Student Name: _____ Date: _____

In Trinidad and Tobago there is a game called **Play Whe**, and great numbers of people play every day. In this game, players place any sum of money on one number from the set 1 to 36. Players can bet on only one number per day. Each day a number is drawn and winners receive 24 times the amount wagered. All other money is lost and there are no consolation prizes. If you wish to pick a number the next day, then you must bet again.



Source: www.nlc.co.tt/home/playwhe.php

Play Whe (traditionally known as **Whe Whe**, an almost identical numbers game) was brought to Trinidad and Tobago by Chinese immigrants. At that time it was known as known as **Chinapoo**, and was a very popular game of chance.

Leading Question

What is the best strategy to maximize profit? Should you play the same number each day or should you vary the numbers?

THE WHE TO PLAY

Student Name: _____ Date: _____

4. If you want to be assured of always making a profit, what changes would you make to your previous model? Predict what your graph will look like if profits are to increase with successive bets.

5. Devise a plan to increase your bet by the same amount each day. How much will you have spent after 5 days in this model? Is this a better method than the previous one from question 2? How much profit will you have if you win on that 5th day?

6. Investigate this plan over a series of successive bets by calculating and recording your profits in a table. If available, use a spreadsheet program to generate the table in which the profit is calculated each day for a period of 48 days. Plot the profit on a graph to observe any trend.

7. If you continue with this plan, will you always make a profit? Assuming you do not win, after how many plays will you choose to discontinue this plan? State why you may wish to stop. Would you stop sooner or later than with the first plan?

THE WHE TO PLAY

Student Name: _____ Date: _____

A sequence of numbers is said to be in an **arithmetic progression** if there is a common difference between consecutive numbers in the sequence. The sequences shown below follow this type of progression:

1, 4, 7, 10, 13, 16, ...

5, 10, 15, 20, 25, ...

If a particular term needs to be predicted, say the 20th term, you do not have to list all the terms. This can be done by observing a pattern and deriving a rule.

8. Consider the first sequence above and observe the table below where T_n is the n th term of the sequence.

$T_1 = 1$	$T_2 = 4$	$T_3 = 7$	$T_4 = 10$	$T_5 = 13$	$T_6 = 16$
1	1+3	1+3+3	1+3+3+3	1+3+3+3+3	1+3+3+3+3+3
1+3(0)	1+3(1)	1+3(2)	1+3(3)	1+3(4)	1+3(5)
$a + d(0)$	$a + d(1)$	$a + d(2)$	$a + d(3)$	$a + d(4)$	$a + d(5)$

Complete the table for the second sequence of numbers with your method for calculating how much you spent in question 5.

$S_1 =$	$S_2 =$	$S_3 =$	$S_4 =$	$S_5 =$	$S_6 =$

9. Use your knowledge of arithmetic progressions to calculate the 20th value in your model from question 5.

THE WHE TO PLAY

Student Name: _____ Date: _____

10. Investigate the effect on the profit when increasing
 - a) the amount of the initial play, a , and
 - b) the fixed difference between successive plays, d .

11. What conclusions can you make when you increase only a while keeping d constant? Observe the trend by determining the profits graphically or algebraically.

12. What conclusions can you make when you vary both a and d ? Observe the trend by graphing the profits.

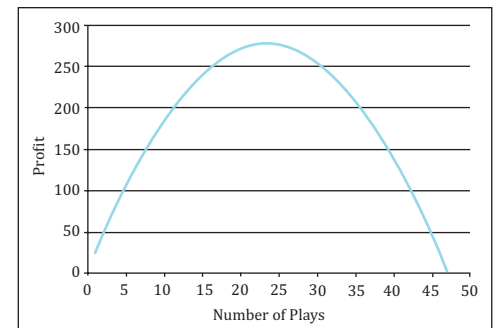
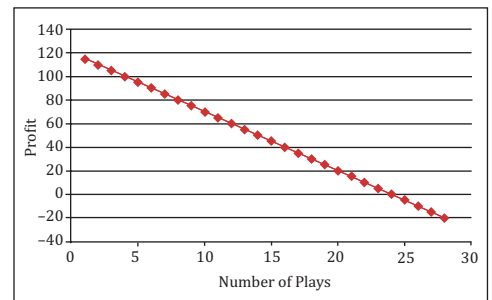
13. What conclusions can you draw about the game? Is it desirable to arrive at a maximum profit quickly?

THE WHE TO PLAY

Teacher's Guide — Possible Solutions

The solutions shown represent only some possible solution methods. Please evaluate students' solution methods on the basis of mathematical validity.

1. \$25, \$50, \$100, and \$150 respectively.
2. Yes for the first three, but not if on the 30th day. Your profits will be \$95, \$70, and \$20 for the first three scenarios, and a loss of \$30 in the final scenario. On day n , a profit of $\$5(24 - n)$ should be expected.
3. Students must recognize that betting the same amount every day will only realize a profit if there is a win within 23 plays. On the 24th play the player breaks even. The scatter plot shows that the profit decreases in a linear fashion (negative slope) and after 24 days, an increase in lost profit will continue to occur. For an increasing profit, students must conclude that the linear model with negative slope is undesirable.
4. A new model must have a positive slope in order to realize an increasing profit with each additional play. A change in strategy must involve moving away from betting a constant amount to an increasing amount. A systematic, rather than random, increase in the amount will enable calculations to be readily made and a new mathematical model to evolve.
5. Answers will vary depending on the amount that students choose to increase their bet each day. If they choose to start with \$1 on the first day and increase their bet by \$1 each day they will have spent \$15 after 5 days. If they won on the fifth day their winnings would be \$120 with a profit of \$105.
6. In the new strategy, students must now investigate how to calculate the potential profit after any number of plays. They should recognize that a mathematical model can be derived to determine the amount of each successive bet.
7. At some point, the plan will have a loss. Depending on the student's model, it will vary.
8. The completed graph will vary with question 5, but a is their starting value, and d is the amount that they increase their bet each day. The bottom line of both tables should be the same, as they are the variable representation of the arithmetic progression.
9. The answer should fit the formula $a + d(19)$.



10-13.

Students may choose to increase a either minimally or substantially. This strategy will produce a model in which the initial profit is high but profits begin to decrease with successive bets (a quadratic function that starts at the maximum and decreases). Students may be questioned on the feasibility of this model. They should conclude that manipulating a is not the option for an increasing profit. In exercising the next option, students may increase d by varying amounts while increasing a by at least d . This strategy will give rise to a model in which the profit increases from the very first play of this phase, increases to a maximum, and then decreases to a point of breaking even before suffering a loss. The pattern is thus similar to earlier examples. Students can now investigate various values of d , while attempting to obtain the maximum profit at around the same value of n as in the earlier example. If a win does not occur before or upon reaching the maximum profit, then a similar exploratory method might need to be employed.

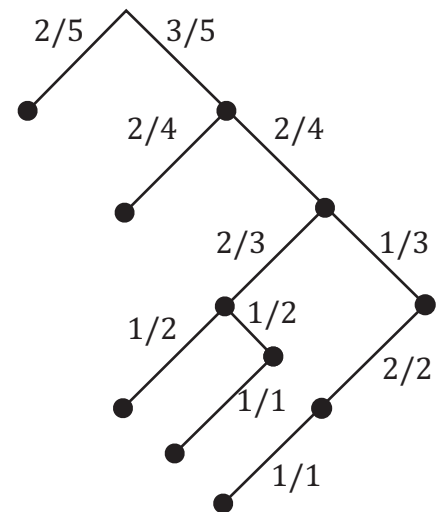
THE WHE TO PLAY

Teacher's Guide — Extending the Model

Since on average $1/3$ of the money bet is lost, Play Whe is probably pretty profitable for the Agency (call it A) that runs it. Schemes that keep increasing the amount of money bet in order to (more than) overcome previous losses eventually flounder because A has greater resources than the individual bettor B. The small probability with which B may lose a huge amount tends to obscure that it is a losing game for B. It may be argued, especially if B bets only small amounts, that the utility of a potentially large win is greater than the utility of a stream of small losses. As many experiments have shown, human utility is not linear, as there is the thrill of participation and of the “Hey, you never know” type advertisement. It can also be argued that a stream of small bets with intermittent wins, such as on a slot machine, is a reasonable price for the diversion provided by the activity.

Play Whe is characterized by being a losing game, although at first glance it looks like there is a winning strategy. The following is an example in the other direction: a winning game which at first glance looks like it must be a loser. You are given an urn which contains 2 red balls and 3 brown ones. If you draw a red ball, you win a dollar, and if you draw a brown ball, you lose a dollar. Do you want to play? Before you say “No!”, consider the precise rules: the drawing is done without replacement, *and* you may stop at any time you wish. The result is that you want to play, and with optimal strategy the expected outcome is 20 cents in your favor.

The strategy is as follows (depicted in the tree diagram to the right, where choosing a red ball is denoted by a move to the reader's left and a brown ball by a move to the reader's right).



- 1) If the first ball you draw is red (with $p = 2/5$), you win a dollar and stop. If the first ball is brown (with $p = 3/5$), you draw again.
- 2) If you now draw a red ball (with $p = 2/4$), you stop, and you have broken even. If the second ball you draw is brown (also with $p = 2/4$), you now know that there are two red balls and one brown ball left and continue.
- 3) If you now draw a red ball (with $p = 2/3$), draw again, and if this fourth draw is red, you are even, and you stop.
- 4) In all other cases, draw all the balls, and lose one dollar.

The probabilities and payouts for this strategy are as follows.

- 1) Stopping here happens with probability $2/5$, and you win one dollar.
- 2) Stopping here has probability $(3/5)(2/4) = 3/10$, and you break even.
- 3) Stopping here has probability $(3/5)(2/4)(2/3)(1/2) = 1/10$, and you break even.
- 4) The probability of the cases so far is $2/5 + 3/10 + 1/10 = 4/5$, so the probability of the other cases, which are the ones in which you draw all five balls, is $1 - 4/5 = 1/5$, and the outcome is a loss of 1 dollar.

Therefore, the expected result of the optimal strategy is $(2/5)(\$1) + (3/10)(0) + (1/10)(0) + (1/5)(-\$1) = \$1/5$, or 20 cents.

The kind of reasoning in this problem has been adapted to find a strategy for deciding when to sell a bond. The key characteristic that a bond shares with the above problem is that there is a known fixed terminal value. If you play our game to the end, you lose a dollar. If you hold a bond to the end, you get its face value. Think of a curve such that if the price goes above that curve, you sell and take your profit. Think also of a second curve such that if the price goes below that curve, you sell and cut your losses.

WATER DOWN THE DRAIN

Teacher's Guide — Getting Started

Diane R. Murray
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Purpose

In this two-day lesson, students will collect data from a water dripping experiment. The data that the students collect will be the basis for estimating how much water is wasted from typical leaky faucets. At the beginning of the lesson, the students are faced with a statistic that states leaky faucets in US homes waste \$10,000,000 worth of water each year. At the end of the lesson, students will have the opportunity to determine what specifications (homes, faucets, drips/minute) result in that amount of money.

Prerequisites

Students need to understand linear equations, graphing techniques, and unit conversions.

Materials

Required: For each group, water, 2 paper cups, 2 paper clips (one small, one large), a ruler, graduated cylinder, and stopwatch.

Suggested: Graphing paper or a graphing utility.

Optional: Internet access.

Worksheet 1 Guide

The first three pages of the lesson constitute the first day's work, which consists mainly of data collection. Separate students into groups of four and have them conduct the experiment. Demonstrate proper use of the materials before students begin and emphasize the importance of accurate measurements when gathering data. Each group should record the data in the table provided and should graph the data. Students will produce plots that will lead to a line of best fit for both the "sink" and "tub" faucets and calculate the slopes of these lines.

Worksheet 2 Guide

The third and fourth pages of the lesson constitute the second day's work in which students will use unit conversions to determine how many are gallons wasted in one day (24 hours) for the sink and tub faucets. The method that the students create will be used to calculate the amount of water wasted in one month (30 days) and one year for both faucets and then applied to national data to determine the amount of water wasted from all households in the country. Questions 11 through 13 are optional since they rely on internet access.

CCSSM Standards

F-LE.1: Distinguish between situations that can be modeled with linear functions and with exponential functions.

F-LE.2: Construct linear and exponential functions, including arithmetic and geometric sequences, given a graph, a description of a relationship, or two input-output pairs (include reading these from a table).

F-LE.5: Interpret the parameters in a linear or exponential function in terms of a context.

WATER DOWN THE DRAIN

Student Name: _____ Date: _____

The US Geological Survey estimates that leaky faucets in US homes waste over \$10,000,000 worth of water each year! Do you have a leaky faucet in your house? How much water do you think is wasted? How much water do you think a leaky bathroom sink faucet wastes compared it to a leaky tub faucet? How would the US Geological Survey reach the conclusion reported?



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Leading Question

How would you design an experiment to estimate how much water is wasted in US homes?

WATER DOWN THE DRAIN

Student Name: _____ Date: _____

In your groups, use the materials given to you by your teacher to create a physical model of the situation of a leaking faucet and a leaking bathtub. Each group should have a water source, 2 different sized paper clips, 2 paper cups, a ruler, a stopwatch, and a graduated cylinder.

It helps to have clearly defined roles for each group member. Who is in charge of which tasks?

1. Describe how you initially plan to set up your model. What jobs do each of the materials play? Might you use other materials not provided by your teacher? If so, what are they?

2. Use the model that you have created in your group to fill in the values in the following table:

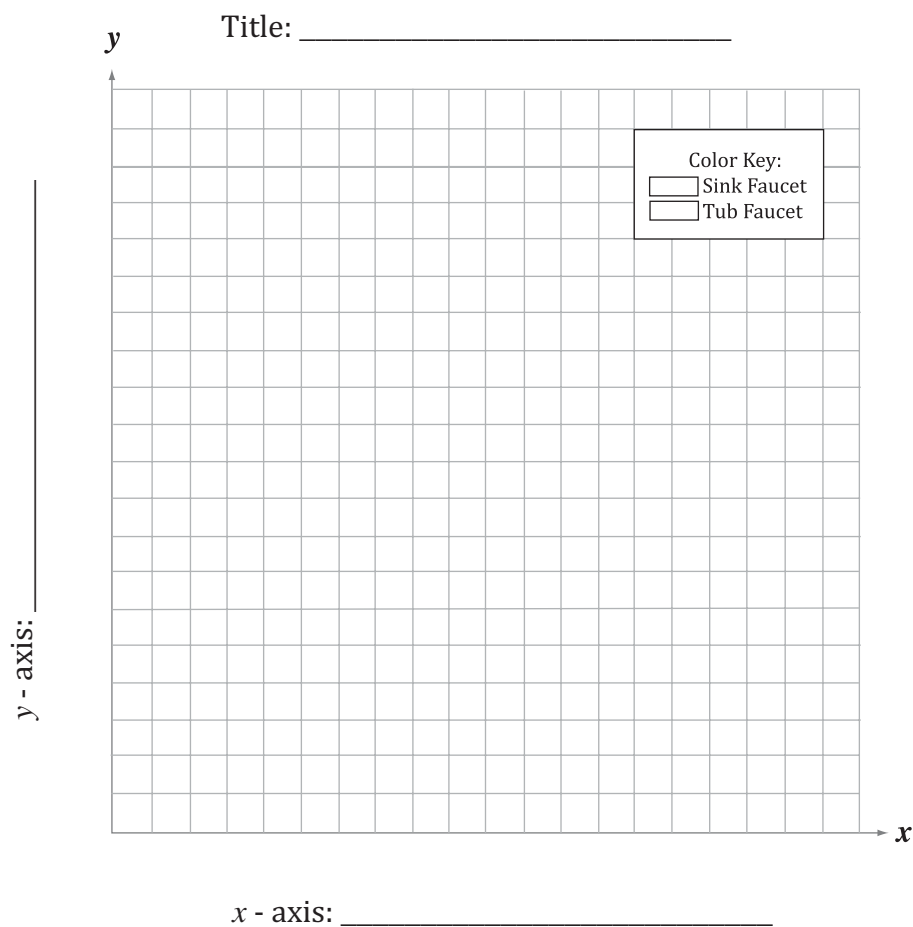
Leaky Bathroom Sink Faucet		Leaky Tub Faucet	
Time (in seconds)	Volume (milliliters)	Time (in seconds)	Volume (milliliters)
10		10	
20		20	
30		30	
40		40	
50		50	
60		60	
70		70	
80		80	
90		90	
100		100	
110		110	
120		120	
Number of drips during first 10 second interval: ____		Number of drips during first 10 second interval: ____	

3. Was your model efficient in its original plan, or did you alter it based on the data you collected?

WATER DOWN THE DRAIN

Student Name: _____ Date: _____

4. Plot both sets of data and draw a line of best fit for both sets of data.



5. How would you determine the slope of a line that seems to fit the points best?

Using your method described above:

a) Find the slope of the best fit line for the Sink Faucet data set: _____

b) Find the slope of the best fit line for the Tub Faucet data set: _____

WATER DOWN THE DRAIN

Student Name: _____ Date: _____

Use the data you collected in the previous day to help answer the following questions.

What do you know about linear functions that could help you answer this questions?

6. How would you write the equation of each best fit line now that you know the slope?

Sink Faucet data set equation: _____

Tub Faucet data set equation: _____

7. Using the best fit line equations, describe a method of estimating the amount of water *in gallons* wasted in one day?

a) Using your method, how much water does the leaky bathroom sink faucet waste in one day?

b) How much water does the leaky tub faucet waste in one day?

8. How much water is wasted in one month (30 days) and one year for both faucets?

9. How many households do you think have at least one leaky faucet? The data from *Census 2010* (<http://www.census.gov/prod/1/pop/p25-1129.pdf>) suggests that there are 114.8 million households in the United States. How much water is wasted in one day from all households in the country?

WATER DOWN THE DRAIN

Student Name: _____ Date: _____

10. A family is going on vacation and accidentally left the leaky bathroom sink and tub drains plugged in.

The sink has dimensions: **Sink depth (in.): 19.125**
 Sink length (in.): 19.125
 Sink width (in.): 8.0

The tub has dimensions: **Tub depth (in.): 8.625**
 Tub length (in.): 60
 Tub width (in.): 30.25

How long will it take to fill the sink completely? The tub?

11. The US Geological Survey has a drip accumulator calculator that can be found online (<http://ga.water.usgs.gov/edu/sc4.html>). How do your estimates compare to their calculations? How many drips/minute did you calculate in your experiments?

Using the drip accumulator calculator, how many gallons per day are wasted in

a) 5 Homes, 2 faucets in each, with 60 drips/minute? _____

b) 10,000 homes, 4 faucets in each, with 20 drips/minute? _____

12. On average 1 gallon of tap water costs 1 cent. How much money is wasted per day from the two examples in question 11?

13. What specifications (households, faucets, drips per minute) would give the estimate that US homes waste over \$10,000,000 worth of water each year?

WATER DOWN THE DRAIN

Teacher's Guide — Possible Solutions

The solutions shown represent only some possible solution methods. Please evaluate students' solution methods on the basis of mathematical validity.

1. Students should be encouraged to try different methods of creating the model given the constraints of each group's specific materials. One possible method for organizing the groups is to assign roles to each group member as follows:
Student 1:
Creates the holes in the paper cups; tests holes for dripping accuracy; counts number of drips during the first 10-second interval.
Student 2:
Start stopwatch when the water begins to drip; alerts group at each 10-second interval.
Student 3:
Fills cup and covers hole with finger until experiment ready to begin; holds cup over graduated cylinder to measure water lost.
Student 4:
Record the amount of water in the graduated cylinder at each 10-second interval.
2. Answers will vary depending on the physical model created by the students but the data should be linear.
3. Efficient models may sometimes be difficult to produce depending on the materials. However, students should think freely about solutions to problems that arise.
4. The tub faucet should have a steeper slope than that of the sink faucet.
5. Finding the average rate of change is an accurate way of determining the slope.
6. Any of the methods of determining the equation of a line work well with this model such as using slope-intercept form. Plotting the points and using a graphing calculator or utility's linear regression can also help to create more mathematically accurate equations.
7. With the x variable representing time (in seconds) in many equations, calculating the number of seconds in a day and then evaluating the functions created in question 6 should give the answer in milliliters. A conversion is necessary to compute the answer in gallons.
8. Multiply the answers in question 7 by the number of days (30) in a month.
9. Estimates on the number of houses with leaky faucets will vary. Determine a reasonable estimate, and then multiply the estimate by the total number of US households, and then by the average amount of water wasted per faucet.
10. The sink has volume of $2,926.125 \text{ in}^3$ and the tub has volume $15,654.375 \text{ in}^3$. Evaluate your equations created in question 6 for $y =$ these volumes. Conversions may be necessary.
11. Answers will vary depending on the models built. The drip accumulator website will give answers of 57 and 76,089 gallons wasted.
12. 57 cents and \$760.89.
13. Answers will vary depending on the number of households/faucets/drips per minute. One solution is 1,000,000 households, with one faucet dripping at 30 drips per minute.

WATER DOWN THE DRAIN

Teacher's Guide — Extending the Model

Sinks and tubs are naturally modeled as if they were boxes (that is, rectangular parallelepipeds), but liquids often come in other containers, which can give rise to questions of some interest. (An amusing sidelight – we won't do any more with it at this point: it is well known that the most economical shape to enclose a given volume is a sphere. So why don't they make spherical milk bottles? Seriously, what criteria should a packaging method satisfy?)

Your typical plastic or paper cup for a drink may not be in the shape of a cylinder, but more likely a section of a cone. Some small paper cups next to a water fountain go all the way down to the vertex of a cone; most have circular cross sections, which are smaller at the bottom than at the top. We would call the shape a **frustum** of a cone. Suppose you want such a cup half-full: how high should you fill it? If you fill up to just half the height, you will clearly have it less than half-full, for every cross-sectional circle below the halfway point in height is smaller than every such circle above the halfway point. So if you want the cup half-full, you will have to fill it to more than half the height. How much more?

Let us assume that the cross-sectional radius grows linearly with height. This is a fairly good model even though it ignores the lip at the top for drinking, and some special circles to make gripping the cup easier. A particular brand of cup (Solo) has a diameter of 6.0 cm on the bottom, 9.0 cm at the top, and is 11.8 cm high. Changing to radii rather than diameters because the familiar formulas are in terms of radii leaves us with the bottom radius $r_0 = 3.0$ cm and the top radius $r_1 = 4.5$ cm. A formula for the radius at height h (where h is between 0 and 11.8 cm) that fits these numbers within the accuracy of 0.1 cm is

$$r = 3.0 + 0.13h.$$

As we said, it won't do to set $r = 3.75$ cm, which is halfway up. It turns out that what we need is a radius of 3.9 cm, and this comes at a height of 6.9 cm. We want to see a convenient way to find this and then to generalize the result. The formula found in solid geometry texts gives the volume, v , of the frustum of a cone of revolution radius, r , at one end and radius, r' , at the other, with a , the altitude, as

$$v = \frac{1}{3} \pi a (r^2 + rr' + r'^2).$$

The typical proof of this requires calculus because it is based on the fact that the volume of a frustum of a cone is the limit of volumes of frustums of inscribed rectangular polygonal pyramids. Those familiar with calculus will recognize the above formula for v more intuitively as

$$v = \frac{a}{(r-r_0)} \int_{r_0}^r \pi x^2 dx.$$

We again see the relevance of integral calculus to the formula for the frustum of a cone. Our question was "When is the cup going to be half full?" At height h along the axis of the cup, the radius of the circular cross-section is $r(h) = r_0 + mh$, where in our problem $r_0 = 3$ cm and $m = 0.13$. Then:

$$r - r_0 = mh \text{ and } \frac{h}{r - r_0} = m.$$

So the volume $v(h)$ from the bottom up to height h is given by

$$v(h) = \pi \int_0^h (3.0 + 0.13x)^2 dx = \frac{\pi}{0.39} [(3 + 0.13h)^3 - 3^3].$$

We want to find h so that this is *half* of the volume of the cup, which is

$$\frac{\pi}{0.39} [(4.5)^3 - 3^3].$$

WATER DOWN THE DRAIN

Teacher's Guide — Extending the Model

So we get

$$(3 + 0.13h)^3 - 3^3 = \frac{1}{2} [(4.5)^3 - 3^3].$$

Approximating the right-hand side of the equation to 59 leads to $h \approx 6.9$ cm.

All this can of course be carried out more generally, but there is an interesting wrinkle at the end. We get:

$$(r_0 + mh)^3 = \frac{1}{2} (r_1^3 + r_0^3),$$

and we can find h by taking cube roots of both sides. But a modeler would also reason as follows: if r_0 equaled r_1 , the h for half a cup would be exactly $(r_0 + r_1)/2$. So it would be natural to want to estimate how far away from $(r_0 + r_1)/2$ the answer is if r_1 does *not* equal r_0 . So let $(r_0 + r_1)/2 = A$ and $(r_1 - r_0)/2 = B$. Then the previous formula becomes

$$(r_0 + mh)^3 = \frac{1}{2} [(A + B)^3 + (A - B)^3] = A^3 + 3AB^2.$$

Remember that we expect B to be smaller (perhaps much smaller) than A . Then we can argue:

$$r_0 + mh = (A^3 + 3AB^2)^{1/3} = A[1 + 3(B/A)^2]^{1/3} \approx A[1 + (B^2/A^2)],$$

where the right-hand side of the last approximate equality is the first two terms of a binomial expansion with exponent $1/3$. So the answer to our “modeler’s question”, namely “what’s a good simple back-of-the-envelope approximation to our answer?” is just $A + B^2/A$. So as a good approximation the cup will be half-full at about B^2/A above the midpoint in height; a simple satisfying answer.

In our problem, A is 3.75 cm and B is 0.75 cm, so that $B^2/A = 0.15$ cm. Summing these two we get 3.9 which is just what we got before!

VIRAL MARKETING

Benjamin Dickman
Brookline, MA

Teacher's Guide — Getting Started

Purpose

In this two-day lesson, students will model “viral marketing.” Viral marketing refers to a marketing strategy in which people pass on a message (such as an advertisement) to others, much like diseases and viruses are spread.

To begin, explain that you are interested in starting your own business and you are researching different marketing strategies to “get the word out.” Viral marketing is one strategy that should be considered. What is viral marketing and what can be said about it *mathematically*?

Prerequisites

Students need have good understanding of exponents and how functions work. The lesson relies heavily on exponential functions to explain how viral marketing works.

Materials

Required: None.

Suggested: Spreadsheet software or a graphing utility.

Optional: Marker chips or index cards (to replicate the passing of a viral advertisement).

Worksheet 1 Guide

The first three pages of the lesson constitute the first day's work. Students are asked to imagine that they are creating an ad campaign for their own business and they want the ad to “go viral.” They will need to model the growth of the ad's viewership for the first week. They will see that physical models become unwieldy very quickly and that a convenient way of organizing this information is necessary. Some students may become frustrated at the rate of growth and may need help understanding that an organized way to construct this model is necessary. In this case, restricting their own models to just a few days should permit them to move on through the lesson.

Worksheet 2 Guide

The fourth and fifth pages of the lesson constitute the second day's work. Students are given a brief description of what an exponential function is and are instructed to modify their models from the previous day to include exponential functions. Particular attention should be paid to the base of the function and its meaning. Students then learn to consider the model they've created, particularly its real-world constraints and the mathematical relationship to other phenomena. Finally, students are challenged to write a business proposal for a marketing strategy. This will reinforce their understanding of the mathematics by sharing their concept with others.

CCSSM Addressed

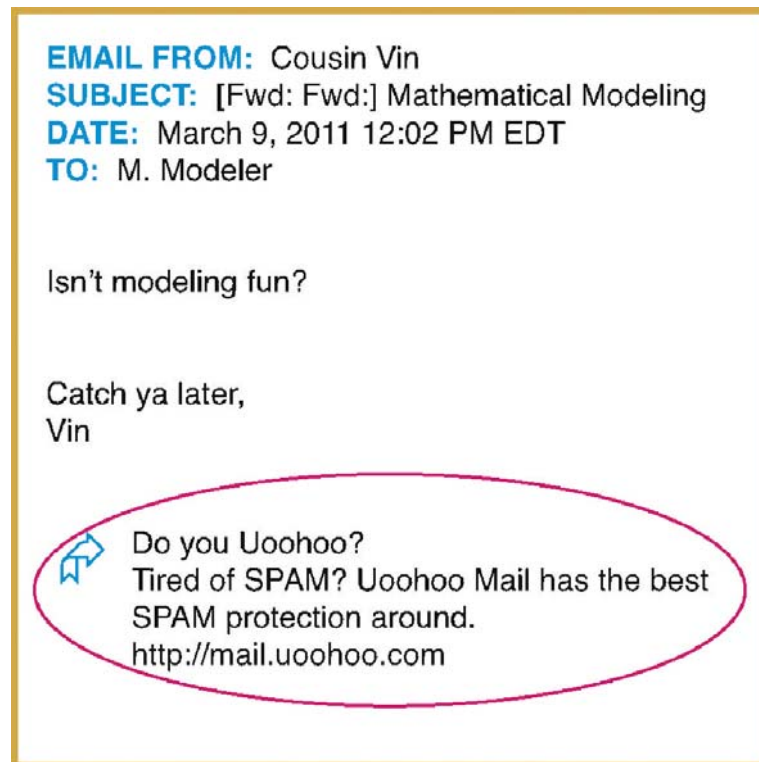
F-LE.1: Distinguish between situations that can be modeled with linear functions and with exponential functions.

F-LE.5: Interpret the parameters in a linear or exponential function in terms of a context.

VIRAL MARKETING

Student Name: _____ Date: _____

“Viral marketing” is an advertising strategy in which people pass on a marketing message to others. For example, when Hotmail first began to offer free email addresses, the following was included at the bottom of every message: “*Get your private, free email at <http://www.hotmail.com>.*” When people received emails from friends and family that were already using Hotmail, many of them would sign up for their own accounts. Later on, these new Hotmail users would send out their own emails, thereby continuing the cycle.



Leading Question

If you wanted to create an advertising campaign for your own business, why might you choose to use viral marketing?

VIRAL MARKETING

Student Name: _____ Date: _____

1. You have created an ad that you want to “go viral” and you show it to several focus groups. Based on their responses, you estimate that the average viewer will send your ad to three other people the next day. If you send the ad to five people on the first day, how many *new* people do you expect will see the ad each day for the first week? How many people *in total* will see the ad each day for the first week?

Is there a way to organize your model to help you see the situation more easily?

2. How can you mathematically describe how many new people will see your ad each day? What about how many people *in total* will see your ad?

3. Use your model to estimate how many people would see the ad in one month. What conclusions can you draw from this estimation?

VIRAL MARKETING

Student Name: _____ Date: _____

4. It turns out that the average viewer forwards your advertisement to six other people the day after they receive it. How does this affect your first week's viewership?

Do you have to make a brand-new model or can you modify the old one?

5. You make a second ad campaign and estimate again that the average viewer will send your ad to three new people the next day. You decide to send the ad to 20 people on the first day. How many new people will see your second campaign each day of the first week? How many people in all will see the ad over the first week?

6. What conclusions can you draw from your answers in questions 4 and 5? What does this say about the mathematics of viral marketing?

Is your model efficient enough to handle this situation? If not, is there an easier way to model the viral growth?

VIRAL MARKETING

Student Name: _____ Date: _____

10. Some more “traditional” forms of advertisement are billboard ads, commercials during a sitcom, and print ads in a newspaper or magazine. Make a model for how the number of people that see your ad changes from day-to-day if you use more traditional forms of advertisement like these. Besides the actual numbers, what is *mathematically* different about traditional ads and viral ads?
11. Design your own campaign! Imagine you’ve started your own business and you need to design an ad campaign that will last for two weeks. Write a proposal to your coworkers about how to carry this out. Don’t forget to include easy-to-understand mathematical explanations for why your campaign will work better than others! (Feel free to be creative and make up expected numbers of people who will see your campaign, but be sure to be reasonable!)

VIRAL MARKETING

Teacher's Guide — Possible Solutions

The solutions shown represent only some possible solution methods. Please evaluate students' solution methods on the basis of mathematical validity.

1. On days 1–7, you expect 5, 15, 45, 135, 405, 1215, and 3645 *new people* to see the ad, respectively. This sums to 5465 people.
2. Use the exponential function $f(x) = 5 \cdot 3^x$, with Day 1 represented by $x = 0$. The values of $f(x)$ when $x = 0, 1, \dots, 6$ represent the new people seeing the ad on a given day. The sum of these first seven values represents the total viewership.
3. Over 30 days, approximately $5.14 \cdot 10^{14}$ people will see the ad. Without the use of mathematics software, students will be unable to do this calculation by hand without an organized model! Considering that there are only about 7 billion people in the world, it seems that there should be more restrictions for this model.
4. On days 1–7, you expect 5, 30, 180, 1080, 6480, 38,880, and 233,280 new people to see the ad. This total equates to 279,935 people.
5. On days 1–7, you expect 20, 60, 180, 540, 1620, 4860, and 14,580 new people, respectively. This total equates to 21,860 people.
6. The conclusion to be drawn shows that the total number of people to see the ad is affected more by the number of people that the ad is passed on to each day. The number of people initially sent the ad does much less to affect total viewership.
7. The base of the exponential function should be representative of how many people the average person shares the ad with.
8. One shortcoming of this unrestricted exponential model is that it grows quickly beyond the population of the planet. Another is that when the real world is considered, people tend to send the majority of their emails within a general group (high schoolers may send the majority of emails to friends in the same high school, most people tend to email to speakers of the same language, etc.). Thus, there will be redundancies in viewers.
9. This is called viral marketing because of the way passing on ads replicates the passing of viral diseases.
10. The traditional forms of advertising only reach the one group of people who see it and viewership generally does not grow. (Mostly commuters on that route will see a billboard, those who watch the show will only see a commercial during a sitcom, etc.)
11. Answers will vary. Mathematical explanations should include the idea of exponential growth, or in the least, continually growing viewership.

VIRAL MARKETING

Teacher's Guide — Extending the Model

Modeling of epidemics is an enticing and potentially fulfilling area of mathematical modeling. For example, trying to understand the relative merits and costs of vaccination and quarantine alone makes it worthwhile. At first glance, the modeling process seems simple: the population consists of the susceptible, the infected, and — for whatever reason — the no longer infectious. But the process of transition from one state to another varies greatly from one situation to another, and can be devilishly hard to model realistically.

Begin with an extremely simple, and simplified, case. Consider a fixed population of N people, consisting only of the susceptible and the infectious. (A better model might take account of the fact that it is possible to be ill but not infectious, and also to be infectious and not know you are ill.) On day n , $n > 0$, you have the number $s(n)$ of “susceptibles,” the number $i(n)$ of *newly infectious*, and the *total number* $t(n)$ of infectious on day n , given by $t(n) = t(n-1) + i(n)$, with $i(1) = t(1) = 1$. A fixed fraction, a , of encounters between the susceptible and the infectious leads to new illness, so that $i(n+1) = i(n) + a \cdot s(n) \cdot t(n)$, and $s(n+1) = N - t(n+1)$. Even this simple model shows the typical S-shaped curve for $t(n)$ versus n .

In the next step towards realism, we introduce a third group, beginning with the assumption of a number $h(n)$ of newly harmless on day n . Each day, a fraction, b , of infectious become harmless, so $h(n) = b \cdot t(n)$. In this model, $i(n)$ has a peak, and it is possible to study whether eventually everyone will be infected, or some fraction will have escaped when the epidemic is over.

So now our equations are

$$s(n+1) = s(n) - a \cdot s(n) \cdot t(n)$$

$$i(n+1) = a \cdot s(n) \cdot t(n) - b \cdot t(n)$$

$$t(n+1) = t(n) + i(n+1)$$

$$s(1) = N, i(1) = 1, t(1) = 1, h(1) = 0$$

$$s(n) + t(n) + h(n) = N$$

In one use of such a model, vaccination would keep $s(1)$ from being all of N , while quarantine would decrease $t(n)$. Looking at the resulting graphs would help to tell how to divide resources between the two.

Modeling the spreading of *rumors* can be done in very much the same spirit, but there are different versions of when an “infectious” person, that is, one who is actively telling the rumor, will stop. A tempting model might assume that two infectious persons meeting and trying to tell the rumor to each other would result in *both* moving into the harmless group: they don't want to be telling stale rumors! What happens when an infectious meets a harmless is debatable. The chance that the infectious would then quit telling the rumor is probably smaller than if (s)he met another infectious. It's interesting to ponder the analogs of vaccination and quarantine in the “rumor” version of the model.

SUNRISE, SUNSET

Teacher's Guide — Getting Started

Edward A. DePeau III
Central Connecticut State University

Purpose

In this two-day lesson, students will examine changes in the average monthly sunlight over the course of a year. They will use actual sunrise and sunset data found on the internet in order to calculate the “length of an average day” for the chosen city. Students will model the data with a sine curve. The model will be interpreted and used to make connections to the real world.

Prerequisites

Students should understand amplitude and period of sine and cosine functions.

Materials

Required: Data form and a graphing calculator or spreadsheet software.

Suggested: Internet access.

Optional: None.

Worksheet 1 Guide

The first three pages of the lesson constitute the first day's work. Students are asked to think about how the length of the day changes throughout the year. It is important that students not only use real data but that they also translate the data into values that will help them to understand the behavior of the model better. By performing the “number of hours” calculations, students are creating a tabular representation for this model. Some students will begin to make strong connections to the periodic behavior while developing the tabular representation while other students will need the graphic representation to understand how the number of daylight hours changed throughout the year. Once the model is created, it is important for students to then begin to analyze the model and connect symbolic representation to the real world.

Worksheet 2 Guide

The fourth and fifth pages of the lesson constitute the second day's work. Students explore the model they've created and use it to make decisions. Finally, they research a city of their choice and must create a model that describes that city's length of day. They should present their findings to the class; presentations should be mathematical and informative about their city's geographic location at the same time.

CCSSM Addressed

F-TF.5: Choose trigonometric functions to model periodic phenomena with specified amplitude, frequency, and midline.

SUNRISE, SUNSET

Student Name: _____ Date: _____

The Earth rotates at a 23.5° tilt from the vertical. As the Earth revolves around the Sun, the amount of sunlight that each location receives changes based on its location and the relative position of the tilt to the Sun. If the Earth wasn't tilted, the amount of daylight at every location would be equal year-round.



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Leading Question

How do the lengths of the days change throughout the year? Is the change constant? Does it matter where you live? Is there any part of the Earth that receives 24 hours of sunlight?

SUNRISE, SUNSET

Student Name: _____ Date: _____

1. If the length of day is defined to be the number of hours of sunlight or the length of time from the sunrise to the sunset,
 - a) Would the number of hours from one day to the next be a constant difference?
 - b) How does the number of hours at the beginning of the year compare with the number of hours at the end of the year?
2. The following table displays the sunrise and sunset times of the 15th of every month in 2010 for Hartford, CT. The data were found on www.sunrisesunset.com.

Month Number	Sunrise Time	Sunset Time	Hours of Sunlight	Month Number	Sunrise Time	Sunset Time	Hours of Sunlight
1	7:16a	4:44p		7	5:28a	8:24p	
2	6:47a	5:22p		8	5:58a	7:52p	
3	7:03a	6:56p		9	6:30a	7:02p	
4	6:11a	7:30p		10	7:02a	6:10p	
5	5:31a	8:02p		11	6:39a	4:31p	
6	5:15a	8:27p		12	7:10a	4:20p	

- a) Complete the table by calculating the number hours of sunlight for the 15th of every month beginning with January (Month Number 1).
- b) What can you say about the length of the day based upon the data in the table?
- c) Use graphing technology to make a scatter plot of the length of day versus the month number.

SUNRISE, SUNSET

Student Name: _____ Date: _____

3. What type of model, if any, do you think would fit the data?

4. What information does the model give you about the length of the day throughout the year? Is there a special feature of your model that indicates the difference in the length of the day over the year at its maximum and minimum?

SUNRISE, SUNSET

Student Name: _____ Date: _____

5. What is the average amount of sunlight that Hartford, CT received per day in 2010?

6. During what day(s) of the year will Hartford, CT have a day where half of the day has sunlight and half of the day does not?

7. How many hours of sunlight will Hartford, CT receive on your birthday?

SUNRISE, SUNSET

Student Name: _____ Date: _____

8. A friend plans to move from Hartford, CT to some other US city. He went onto the website www.sunris-sunset.com and created models for various cities around the US he is interested in moving to. One particular model that he developed for a certain city was $f(x) = 6.6\sin(0.508)(x - 3.1299) + 12.2$, where x is the number of the month in the year and $f(x)$ is the number of hours of sunlight. Determine a city for which the model might be appropriate. Explain why the determined city fits the model.
9. Are there businesses that might benefit from these models? What kinds of businesses would benefit? How might a business benefit from knowing a model like the one you created?
10. Pick your favorite city and collect data on the length of the days for a certain year. Create a model of the data and explain how it can be used to benefit you, an organization, business, or government agency. Present your findings to the class.

SUNRISE, SUNSET

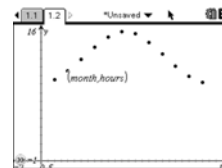
Teacher's Guide — Possible Solutions

The solutions shown represent only some possible solution methods. Please evaluate students' solution methods on the basis of mathematical validity.

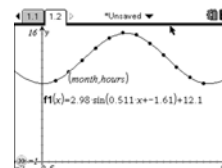
- 1 a. No.
- 1 b. It would be approximately the same amount. The number of hours of sunlight received on December 31st is not much different than the number of hours of sunlight received on January 1st.

Month Number	Sunrise Time	Sunset Time	Hours of Sunlight	Month Number	Sunrise Time	Sunset Time	Hours of Sunlight
1	7:16a	4:44p	9.46	7	5:28a	8:24p	14.93
2	6:47a	5:22p	10.58	8	5:58a	7:52p	13.9
3	7:03a	6:56p	11.88	9	6:30a	7:02p	12.53
4	6:11a	7:30p	13.32	10	7:02a	6:10p	11.13
5	5:31a	8:02p	14.52	11	6:39a	4:31p	9.87
6	5:15a	8:27p	15.2	12	7:10a	4:20p	9.17

- 2 a.
 - b. As the year progresses the average number of hours of sunlight per month increases at a non-constant rate until June and then the average number of hours decreases at a non-constant rate. Again, the average number of hours of sunlight in December is similar to the average number of hours in January.
 - c. The following graph was created using a TI-nspire.



3. The model needs to be periodic. One possible model is shown.



4. The model shows that the length of day is periodic throughout the year. The amplitude helps determine the difference in the length of day between the longest and shortest days of the year.
5. 12.1 hours.
6. $x = 3.09$ months and $x = 9.36$ months. The corresponding dates are March 2nd and September 11th.
7. Answers will vary depending on the student's birth month.
8. Answers will vary. One possible city is Juneau, AK as the model has as much as 18 hours of sunlight and as few as 6 hours of sunlight.
9. Answers will vary; one possible business would be gardening centers.
10. Students will present their findings based on their own data.

SUNRISE, SUNSET

Teacher's Guide — Extending the Model

Another question about the Hartford length-of-day data is when the length of the day is changing most rapidly. In June, the days are longest but the length is changing very little from one day to the next. Then in December, the length of the day is small but again it is changing very little from day to day. In the latter part of March, on the other hand, the number of hours of sunlight changes upwards of 3 minutes per day at its maximum. Later on, students will see this as characteristic of an inflection point in the curve.

To look for other sets of data with such clean periodicity and little added noise, other geophysical phenomena are tempting. Data on tides will be good if you are near an ocean; the visible fraction of the moon is periodic and you'll have fun deciding how to define and measure it.

Strictly human creations that can provide nice periodic data include Ferris wheels and bicycles. If you are watching a Ferris wheel from not too far away, you can take a sequence of photos or make a short movie, pick a particular spot (for example, by its décor), and then analyze the height of that spot as a function of time. You have to pick your Ferris wheel carefully, of course: sometimes the famous wheel in the Prater in Vienna keeps stopping so that every car can get a view from the top and then allows you just *one* time around before they make you get out again! (What a disappointment!)

Other physical phenomena may be basically periodic, but may have decreasing amplitude. If you plot the oscillations of a tuning fork, you will get nice periodic data but the volume of sound decreases with time. You will obtain data of similar shape if you measure the height of a basketball as it bounces in as close to one spot as you can make it. Set up your graphing calculator's motion sensor above the ball and record the distance downward to the top of the ball. You will have fun interpreting the results. (Why? If you subtract the diameter of the ball from its computed height, you may get *negative* minimum heights for the bottom of the ball and will have to explain them!)

Human business activities dependent on the length of daylight are likely to have periodic aspects but there will be additional considerations of a nature different from decreasing amplitudes. Monthly housing starts, if you are not too far south in the country, have the basic periodicity of daylight but there will be a long-term trend depending on the local economy that you will need to identify and separate out as an added slowly varying function. Nowadays, this is unlikely to be linear! A trend in housing that starts with data from, say, 10 years ago, will probably be more nearly linear, but the difference between "then" and "now" may be painful to discuss. Daily temperature in your community – be it maximum, average, or minimum – again inherits its basic periodicity from the length of daylight, but the storage of heat in the ocean changes where the annual peaks and valleys occur. Again, if your data show evidence of global warming, or if they don't show such evidence, you might need to be prepared for non-mathematical aspects to the discussions.

SURVEYING THE ANCIENT WORLD

Heather Gould
Stone Ridge, NY

Teacher's Guide — Getting Started

Purpose

In this two-day lesson, students will construct and use a simple version of an ancient tool called an **astrolabe**. This tool measures the angle between the tool and the horizontal plane. It was used frequently by ancient surveyors, engineers, astronomers, and seafarers to compute angles and heights.

To introduce the lesson, explain the use of the astrolabe to students and have them imagine that they are ancient surveyors trying to measure the heights of mountains. The astrolabe only measures angles, though. How could ancient surveyors complete their task?

Prerequisites

Understanding of the properties of similar triangles and some knowledge of trigonometric ratios are helpful but not necessary. Students must be able to make accurate scale drawings and convert between two scaled measurements.

Materials

Required: Simple astrolabe copies (given on next page), straws, string, washers (to serve as a plumb bob), tape, metric rulers or measuring tape, protractors, calculators, mural of mountain range to post on a tall wall (made by teacher).

Suggested: Internet.

Optional: None.

Worksheet 1 Guide

The first three pages of the lesson constitute the first day's work. Students must build a simple astrolabe using the image on the next page. An explanation of how to make and use a simple astrolabe can be found online at http://cse.ssl.berkeley.edu/AtHomeAstronomy/activity_07.html. To save time, the teacher might choose to make several astrolabes in advance. To begin the work of the lesson, students will determine what information they will need to find the height of an object. They will measure the angles found from a certain distance (3 meters is recommended) to the tops of several "mountaintops" with known heights on a wall mural. Answer 2 in the Possible Solutions guide can be used as a suggestion for the heights of the mountaintops. Students will look for mathematical patterns in their findings as well as in scaled-down versions of the situation. Finally, they create a mathematical model to find unknown heights based upon scaled versions of triangles.

Worksheet 2 Guide

The fourth and fifth pages of the lesson constitute the second day's work. Students are introduced to trigonometric ratios and refine their previous model. It is not always easy to find the distance to a point below the top of an object and the students create a model using two measurements to account for this. Finally, they try to apply their model to more situations.

CCSSM Addressed

G-SRT.8: Use trigonometric ratios and the Pythagorean Theorem to solve right triangles in applied problems.

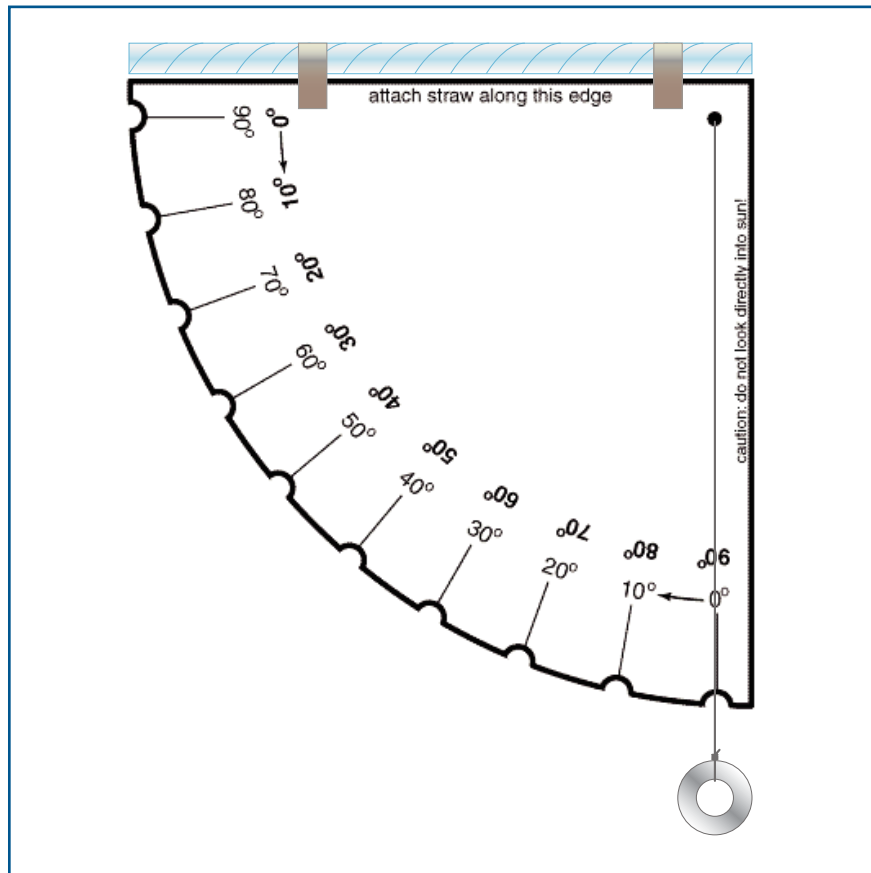
G-MG.3: Apply geometric methods to solve design problems.

SURVEYING THE ANCIENT WORLD

Student Name: _____ Date: _____

A surveyor's job is to determine the position of places and objects around the world accurately. Surveyors assist in the creation of maps and boundaries. The accuracy of their results is extremely important to scientists, engineers, and even property owners! In the ancient world, surveyors were just as important and they had to do their job well with only ancient tools. One such tool is an **astrolabe** which helps determine the height of an object by measuring the angle formed between the object and a horizontal plane (usually the ground or in the case of an astronomical object, the horizon).

The picture below will help you make a simple astrolabe.



Derived from a picture at http://cse.ssl.berkeley.edu/AtHomeAstronomy/act07_astrolabe.html

Leading Question

How did ancient surveyors use an astrolabe to determine the height of various land features?

SURVEYING THE ANCIENT WORLD

Student Name: _____ Date: _____

1. How do you think knowing angles helps determine unknown heights? What other information might you need to help determine height? Draw a picture describing the situation; label what you know, need to know, and what you're trying to find.

What kinds of measurements do you think ancient surveyors could obtain? What other types of tools do you think they could use?

2. Experiment: Build a simple astrolabe and use it to help you determine the various angles between the ground and several "mountaintops" with known heights, each from the same distance, for example, 3 meters away. Record your findings in the table below.

(Actual) Distance from "Mountain"	(Actual) Height of "Mountain"	Angle Found

3. What patterns do you notice in your findings from question 2? Is there a mathematical relationship to describe the pattern? If so, what is it?

SURVEYING THE ANCIENT WORLD

Student Name: _____ Date: _____

4. It is often helpful in modeling to consider the problem on a smaller scale. Draw and examine a scaled-down version of the situation using the distance from the mountain and angles found to guide you. Do you notice the same patterns?

Distance from Mountain	Angle Drawn	Height of Mountain

5. Can you create a mathematical model to describe how to find the unknown height of a mountain?

How can you use your knowledge of similar triangles to help you create a model?

6. Try experimenting using different distances from the mountaintops with known heights. Does your model still work? If not, try to modify your model so that it works in all cases.

SURVEYING THE ANCIENT WORLD

Student Name: _____ Date: _____

7. You may have noticed that a major problem with using an astrolabe to determine heights is that different people will often find different angles to the top of an object even though they were standing at the same point. Why might that happen? How can the problem be fixed?

Trigonometric ratios help determine the lengths of the sides of a right triangle. The ratios of the lengths of adjacent sides in each of two similar triangles are equal. **Tangent** relates the lengths of the leg opposite an angle to the length of the leg adjacent to that angle.

8. How can the tangent of an angle be used to determine unknown heights? Describe the model you think ancient surveyors used. Is it similar to the model you created in question 5?

9. It is usually difficult to determine the exact horizontal distance from a person using an astrolabe to a point directly below the highest peak (or other highest entity) because that point is usually inaccessible within the object being measured. For example, in the picture below, the point that you'd need to measure from is inside the volcano! Do you think it is possible to use two measurements to determine the height of the peak of the volcano? Modify your model to describe how this could be done.



SURVEYING THE ANCIENT WORLD

Student Name: _____ Date: _____

10. Construction of the Leaning Tower of Pisa began in 1173. It began to lean because it was built on very soft ground and one side began to sink. It is very important to take accurate measurements to monitor the tower and ensure that it doesn't topple! How could Pisans in pre-Renaissance Italy use your model from question 9 to monitor the tower?

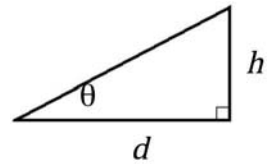
11. An important part of the modeling process is determining where else your model can be used. For example, if you *knew* the height of something you were approaching that was far into the distance, could you use your model to determine anything important? If so, how?

SURVEYING THE ANCIENT WORLD

Teacher’s Guide — Possible Solutions

The solutions shown represent only some possible solution methods. Please evaluate students’ solution methods on the basis of mathematical validity.

- Knowing an angle along with one other side helps to determine heights because similar triangles always have the same angles. The side with known length can be used to determine the “scale” of the length of the side that is unknown. The length of at least one other side needs to be known; usually the distance from the object in question to a person with an astrolabe is sufficient. The variables in question are h , the height of the object to be determined, d , the horizontal distance from the person measuring to a point directly below the top of the object, and θ , the angle measured between the horizontal and the top of the object.



Distance from “Mountain”	Height of “Mountain”	Angle Found	Distance from Mountain	Angle Drawn	Height of Mountain
3 m	0.80 m	15°	9 cm	15°	2.4 cm
3 m	1.73 m	30°	9 cm	30°	5.2 cm
3 m	3.00 m	45°	9 cm	45°	9.6 cm
3 m	5.20 m	60°	9 cm	60°	15.6 cm
3 m	11.20 m	75°	9 cm	75°	33.6 cm

- Peaks of taller mountains make larger angles from the same distance away.
- The pattern found here is similar: Larger angles yield larger heights. An astute student will notice that the ratio of the heights to the distances will be the same for the same angles in questions 2 and 4.
- Without knowledge of trigonometric ratios, a student might choose to use the measurements found on the scaled version of a triangle and then multiply by the correct factor to “scale up”. Thus, if the actual distance is d , the angle found between the horizontal and the mountaintop is θ , and the height to be determined is h , and the scaled triangle has base d_1 , an angle between the base and the top of the scaled object θ , and height h_1 (found using a ruler), and $d = kd_1$, where k is some constant, then $h = k \times (h_1/d_1)$.
- The model given works in all cases.
- People with different heights will measure different angles. This can be adjusted by measuring the distance from the ground to the person’s eye and adding it to the distance computed.
- $d \tan \theta = h$. Since tangents are the same for any right triangle with angle θ , then tangent tables can be used and only simple calculations are needed. Ancient people would only need to find the tangent of each angle once and then the results could be stored in a table and this table could be used instead of computing the necessary ratio each time. This is essentially the model described in question 5 since $h_1/d_1 = \tan \theta$.

SURVEYING THE ANCIENT WORLD

Teacher's Guide — Possible Solutions

9. Two measurements can be taken from a known distance apart, for example, 500 feet. Let the first measurements taken be d_1 and θ_1 and the second measurements taken (500 feet back) be d_2 and θ_2 . We know $d_2 = d_1 + 500$. Then since the height, h , is the same for both measurements, we can find d_1 by the algebraic reasoning $d_1 \tan \theta_1 = d_2 \tan \theta_2 = h$, which implies $d_1 \tan \theta_1 = (d_1 + 500) \tan \theta_2$. Solving for d_1 and then replacing it in the equation $d_1 \tan \theta_1 = h$ gives the final solution. This solution can be shortened by using cotangents of the same angles if students are comfortable with the reciprocal trigonometric functions.
10. The same model from above could be used at different times, perhaps yearly. Even a slightly lower height found could indicate that the tower is sinking.
- 11 In this case, the height remains constant but the distance from the object is changing. Two measurements taken over a specified amount of time could help determine both the distance traveled as well as average speed. Immigrants arriving by boat to Ellis Island may have found it entertaining to determine how far they were and how quickly they would be arriving in their new home by using the height of the Statue of Liberty, for example. They would have to make further calculations based on the distance between Liberty Island and Ellis Island and the angles between the two islands and the boat.

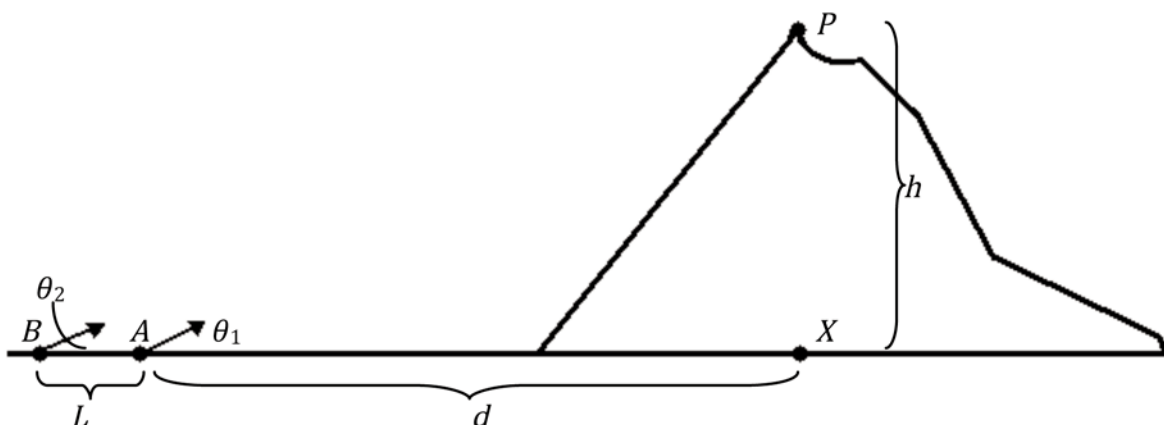
SURVEYING THE ANCIENT WORLD

Teacher's Guide — Extending the Model

In the applications of trigonometry to surveying, measurement error is a major concern. Neither distances nor angles can be measured with perfect accuracy. Is there anything you can do to give you some protection against the effects of measurement inaccuracy? It turns out that the geometry of the relative positions of the measurement devices and the to-be-measured quantities can affect how sensitive the results will be to the accuracy of the measurements.

An example of this shows up when you think further about question 9, computing the height h of a volcano peak P without having to get close to it. The question assumes you have a base camp at A from which you have a clear view of P . You now look for a second location B with a clear view of P such that the line joining A and B is horizontal and the plane containing A , B , and P is vertical.

Let X be the projection of P onto the line containing A and B . Let θ_1 be the degree measure of angle PAX , θ_2 the degree measure of angle PBA , d the distance between A and X , and L the distance between B and A . (See the picture below, not drawn to scale.) We can measure the two angles θ_1 and θ_2 and the distance L , we cannot measure the distance d , and we expect to compute the height h . The location of A determines θ_1 , the location of B determines L and θ_2 , and we use L , θ_1 , and θ_2 to compute h . Given the location of the base camp A , which determines θ_1 , we will see that there is a unique value of L so that the computation of h is least sensitive to the accuracy with which θ_2 is measured.



Let us first try a numerical example. Let θ_1 be exactly 30° and let L be exactly 500 feet. Suppose that the true θ_2 is 14.85° . Our formulas are

$$\cot \theta_1 = \frac{d}{h} \text{ and } \cot \theta_2 = \frac{L+d}{h}.$$

We subtract the first formula from the second, rewrite, and obtain

$$h = \frac{L}{\cot \theta_2 - \cot \theta_1}. (*)$$

This simple formula for h can be rewritten as

$$h = L \frac{\sin \theta_1 \sin \theta_2}{\sin \theta_1 \cos \theta_2 - \sin \theta_2 \cos \theta_1} = L \frac{\sin \theta_1 \sin \theta_2}{\sin(\theta_1 - \theta_2)}. (**)$$

We obtain that h is 245.2 feet.

What if there is an error of $\pm 1^\circ$ in our value of θ_2 ? If θ_2 is 13.85° , then h becomes 215.2 feet, and if θ_2 is 15.85° , then h becomes 279.3 feet.

SURVEYING THE ANCIENT WORLD

Teacher's Guide — Extending the Model

If we use a smaller value of L , for example $L = 200$ feet, then the correct θ_2 would be 21.43° in order to produce the previous value of h . If *this* value of θ_2 were in error of $\pm 1^\circ$, then the values of h would become 210.0 feet and 289.6 feet, respectively. If we use a larger value of L , for example $L = 1250$ feet, then the correct θ_2 would be 8.33° in order to produce the previous value of h . If *this* value of θ_2 were in error of $\pm 1^\circ$, then the values of h would become 206.9 feet and 287.1 feet, respectively.

The changes in sensitivity when L goes from 200 to 500 to 1250 feet are not spectacular, but they are not monotone either! The sensitivity is *less* at $L = 500$ than at either 200 or 1250 feet! But why is this so? We will first give an instinctive reason and then give an analytic one. Suppose L is small. Then B is very close to A , therefore θ_2 must be close to θ_1 , and even a small error in θ_2 will cause havoc in the computation. If, on the other hand, L is very large, then θ_2 itself must be very small, and again a small error will destroy the computation. So it makes sense that there should be a value of L in between that causes the least trouble.

That's the instinct. Now let us use calculus and see what we get. Let's take the equation (**), find the partial derivative of h with respect to θ_2 , and then divide by h . When multiplied by a small change, $\partial\theta_2$, this will give us the corresponding relative change $\partial h/h$. We find that

$$\frac{1}{h} \frac{\partial h}{\partial \theta_2} = \frac{\sin \theta_1}{\sin \theta_2 \sin(\theta_1 - \theta_2)} \quad (***)$$

We see that if L is small, so that θ_2 is very close to θ_1 , the denominator is almost 0, and the answer is very large. We see that if L is very large, so that θ_2 is very close to 0, the denominator is almost 0, and the answer is very large. In between, there is a value of θ_2 that will maximize the denominator, and that's the value we want. Your instinct will tell you – and calculus will verify it – that this happens when $\theta_2 = \theta_1 - \theta_2$, that is, when the angle at B is half the angle at A . But this means that the triangle is isosceles, namely the length of BA equals the length of AP . And this is just about what happens when $L = 500$ feet!

How do we know? Well, the problem was made up with $\theta_1 = 30^\circ$ so its sine is 0.5, and therefore if h is 245 feet, then AP is almost 500 feet. So it all checks! Moreover, if you substitute the numbers for $L = 500$ into the equation (***) and multiply by a radian value corresponding to $d\theta_2 = 2^\circ$, you get $dh = 63.6$ feet, which is almost the 64.1 feet we computed above. Isn't mathematics wonderful?

PACKERS' PUZZLE

Teacher's Guide — Getting Started

Kai Chung Tam
Macau, People's Republic of China

Purpose

In this two-day lesson, students consider ways to estimate the number of spheres that will fit within a container. They also will try to pack as many as possible into differently shaped containers.

The objective of this lesson is to have students use geometric solids so that they can solve basic packing problems that arise in the real world.

Prerequisites

It is assumed that students are familiar with the calculation of area and volume of various shapes. Other geometrical concepts related to circles, such as radius, diameter, and tangent lines, are also relevant. Informal exposition of rigid motions (parallel translation, rotation, and reflection) is preferred.

Materials

Required: Calculator; circular tokens of various sizes (e.g., pennies, bottle caps, checkers), and two-dimensional “containers”. As a preliminary step, teachers need to prepare photocopies of 3 shapes (squares, circles, and equilateral triangles) of three different sizes each. Be sure to note the measurements (sides and radii) of each of these shapes and for each token for students to make proper calculations.

Suggested: None.

Optional: Digital scale. A jar of candy or any container of identical objects.

Worksheet 1 Guide

The first four pages of the lesson constitute the first day's work. Initially students can work on the first two pages individually, but for the next two pages, they should to be organized into groups. Each group will be provided tokens and shapes (containers) to model the orange packing situation. By combining different ideas that the students came up with before they were separated into groups, they can fill out the table provided on the third page of the lesson and answer the questions that follow.

Worksheet 2 Guide

The fifth and sixth pages of the lesson constitute the second day's work in which students should realize that a dense packing is wanted. After a discussion of the density and the unit of a regular arrangement in the plane, students will calculate the theoretical density of rectangular and hexagonal arrangement and compare the result with the first worksheet. Finally, students will think about the extension to three dimensions.

CCSSM Addressed

G-GMD.3: Use volume formulas for cylinders, pyramids, cones, and spheres to solve problems.

G-MG.1: Use geometric shapes, their measures, and their properties to describe objects (e.g., modeling a tree trunk or a human torso as a cylinder).

G-MG.2: Apply concepts of density based on area and volume in modeling situations (e.g., persons per square mile, BTUs per cubic foot).

G-MG.3: Apply geometric methods to solve design problems (e.g., designing an object or structure to satisfy physical constraints or minimize cost; working with typographic grid systems based on ratios).

PACKERS' PUZZLE

Student Name: _____ Date: _____

At the Orange Festival right after a bumper crop, a farmer invites guests from all over the town. He shows the guests a full box of randomly arranged oranges, stating that anyone who could guess the exact number of oranges can take home as many oranges as he or she can carry.



© Wilfred Stanley Sussenbach | Dreamstime.com

Is there a difference between the randomly packed oranges on the left and the regularly arranged oranges on the right?

If you cannot just pour out all of the oranges and then count them one by one, what technique would you use to determine correct the number of oranges in the box?

Leading Question

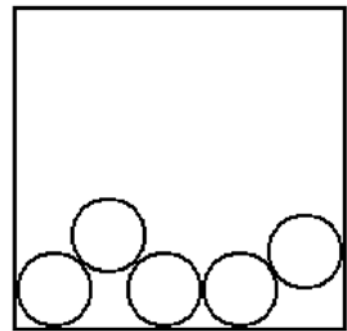
How can you determine the correct number of oranges that are in the container?

PACKERS' PUZZLE

Student Name: _____ Date: _____

1. Before you can answer the question, you need to make some assumptions to think more effectively. One assumption you can make is that all the oranges are spheres. What other assumptions could you make in your model that might not be true in the real world, but are basically useful in creating a mathematical model?

2. Often it is simpler to look at an easier question before trying to attempt a difficult one. In a two-dimensional model, containers become planar. For example, they can be rectangles or triangles. Oranges become circles, which cannot intersect with each other or with the container. What methods might you use to estimate how many circles can be packed into a box, without direct counting? Describe how one of your methods works using words and mathematical notation.



3. How could you use your knowledge of the area of circles to determine the maximum number of circles that can fit into your container?

PACKERS' PUZZLE

Student Name: _____ Date: _____

With your group, use the containers (shapes) that your teacher has provided and fill them with your oranges (tokens). Try different size shapes and tokens. For each trial, choose one shape and one type of token, then try to fit the tokens into the shape. Describe how you fit them in, and fill in one row of the following table. Try to have five unique trials.

Group Names:			Shape of Container:					
#	Side or radius of container (cm)	Capacity of container C (cm ²)	Radius of token r (cm)	Area of token (cm ²)	Maximum number N'	Actual token fit N	$N' - N$	Density $N\pi r^2/C$ (%)
1								
2								
3								
4								
5								

4. How did you fill in your tokens in each case?

5. What accounts for the difference $N' - N$?

6. How could you improve the estimation N' , so that $N' - N$ becomes smaller?

When you propose your way of estimating N' , think of question 5.

PACKERS' PUZZLE

Student Name: _____ Date: _____

7. The density column calculates the true total area of the tokens, which is $N\pi r^2$, divided by the capacity, C , of the shape. What kind of arrangement would give you a higher or lower density, a random arrangement or a regular arrangement? Why?

8. What other ways of calculating the number of oranges might exist? How do you think the farmer knows the number? Do you think he actually counted all of them?

How might measuring the weight of an orange help you to determine how many are there?

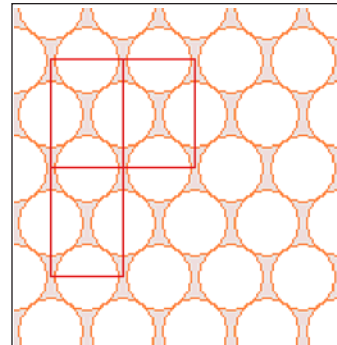
9. Should your answer be a whole number? Explain your reasoning.

PACKERS' PUZZLE

Student Name: _____ Date: _____

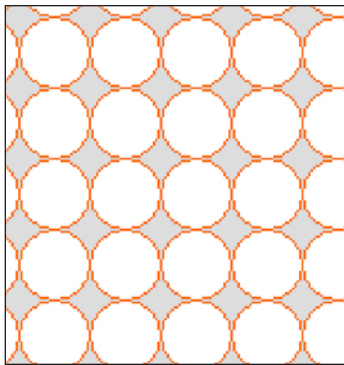
Recall from the previous lesson that the prize for winning was that you got to take as many oranges as you could carry. Suppose that you have already won the prize and the farmer offers some boxes to use to pack oranges.

A **regular arrangement** is one that can repeat indefinitely and looks the same wherever you see it. More precisely, there is a **unit** of arrangement so that you can do parallel translations to repeat the pattern in any direction. The figure on the right shows a regular arrangement, and indicates three copies of the unit. Using just one unit repeatedly, you can extend the picture as far as you want.

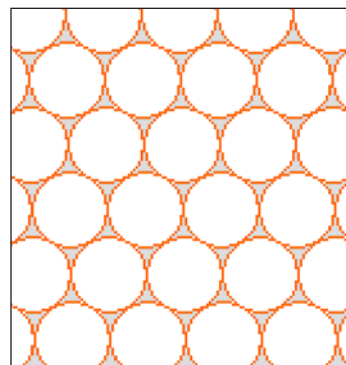


10. Find and draw a unit in each the following two arrangements.

A:



B:



11. If you have a container, **density** is the area used divided by the total area of the container. Find the density of the two units that you have chosen for arrangement A and arrangement B, as if the unit is a container.

PACKERS' PUZZLE

Student Name: _____ Date: _____

12. How do the two densities differ? Can you say that the density within one unit represents the overall density? Why or why not?
13. Compare the results to your classmates' and look at the unit that they have chosen. Did you choose the same unit? Did you get the same density?
14. The Arrangement A is a **square arrangement**, while arrangement B is a **hexagonal arrangement**. Do you understand why they are named this way? Why do you think they were named this way?
15. If you have enough identical spheres (e.g., oranges, gumballs, baseballs), try to pack them regularly into a container for which you know the volume. Knowing what you now know about density, what are possible arrangements? What is the density of each arrangement according to experiment or calculation?

PACKERS' PUZZLE

Teacher's Guide — Possible Solutions

The solutions shown represent only some possible solution methods. Please evaluate students' solution methods on the basis of mathematical validity.

1. Some examples: all oranges are identical objects, all of them are spheres, the spheres and the container should not overlap each other, and the spheres and the container are rigid.
2. Any method will do, but good methods will have some form of organization to them. For example, organize the tokens in rows and stack the second row on top of the first, so the circles are notched together, and the height is minimal.
3. Students should use the formula for the area of a circle, $A = \pi r^2$, and the area of the shape (capacity) to determine an upper bound for the number of oranges that can fit inside the shape by dividing the capacity by the area of one token.
4. See question 2 for an answer. Since the students are now in groups, combinations of methods might have also been created.
5. The space not occupied by the circles accounts for the difference. The more unused space there is, the larger the difference.
6. If it is a regular arrangement, the denominator can be changed to the area of a "unit containing one circle". If it is a random arrangement, it is not very easy to estimate well by this method; however, in three dimensions there is a way to estimate the volume of the unused space. Use any liquid to fill it up the container to capacity and then measure the amount of liquid used! These are not advanced methods so students should be motivated to find one.
7. Some students might pack tokens into the shape randomly while others might do a regular arrangement, so the "density" will vary. Yet, if we fix one arrangement of packing (random, squared, hexagonal), the density has only a little difference.
8. One other way to calculate is dividing the total weight by the weight of one orange.
9. According to our assumption "that all oranges are equal", the quotient should be exactly the same as the number of oranges, but in reality, sizes vary.

10–15.

In a square arrangement, each circle touches four other circles; in a hexagonal arrangement, each circle touches six others. We use red lines to draw a unit. On the left, all these rectangles are correct units. Spheres have diameter equal to 1 cm. In the two squared ones, the total area of a unit is 1 cm^2 . The used area is $\pi \cdot (0.5 \text{ cm})^2$ therefore the density within each unit is

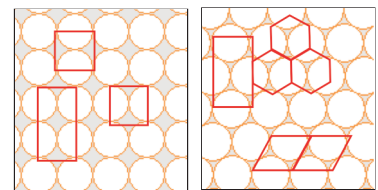
$$\frac{\pi(0.5 \text{ cm})^2}{1 \text{ cm}^2} = \frac{\pi}{4} \approx 78.54\%. \text{ The larger rectangle}$$

gives the same density. On the right, the rectangle, parallelogram, and hexagon are all correct "units". For the rectangle, total area = $(1 \text{ cm}) \cdot (\sqrt{3} \text{ cm}) \approx (\sqrt{3} \text{ cm}^2)$, and the used area = area of

2 circles = $\frac{\pi}{2} \text{ cm}^2$, therefore the density = $\frac{\pi}{2\sqrt{3}} \approx 90.69\%$. Using the parallelogram, the total area

becomes $(1 \text{ cm}) \cdot (\frac{\sqrt{3}}{2} \text{ cm}) = \frac{\sqrt{3}}{2} \text{ cm}^2$, and the used area equals exactly one circle, therefore the density

is $\frac{\pi}{4} \div \frac{\sqrt{3}}{2} = \frac{\pi}{2\sqrt{3}}$, the same as before. The hexagon also gives the same result.



PACKERS' PUZZLE

Teacher's Guide — Extending the Model

A fascinating and far-reaching extension concerns fruit in a supermarket. The weekly ads often give a size, like “15-size cantaloupe”. This applies to any fruit that is large enough to buy individually, like grapefruit, pears, or lemons, but not to fruit like blueberries, cherries, or currants. Students could investigate what these numbers mean. Is a 12-size cantaloupe smaller or bigger than a 15-size cantaloupe? What do these numbers have to do with the way cantaloupes are packed and shipped? In fact, you could base a goodly portion of a geometry course on the desire to understand the answers to such questions.

A quick beginning of an answer is that cantaloupes, for example, most commonly come in one of the following sizes: 9, 12, 15, 18, 23, and 30. The smaller the number, the larger the cantaloupe. Why? The numbers indicate how many will fit into a standard 40-pound case or shipping box, so 9-size is the largest. The shapes of standard boxes are carefully chosen so that the right number of melons of any one size will fit comfortably but with very little wasted space into the same standard-size box. Avocado sizes come in all multiples of 4 from 20 to 40, and then 48, 60, 70, 84, and 96. The most common pear sizes are 70, 100, 150, and 215. Boxes are marked on the outside with the size numbers of the contents. If you know someone in the fruit and vegetable department of your supermarket, for example, have a look at the clever shapes of the boxes which are adaptable to contents of different sizes.

You can begin thinking about boxes for packing fruit by thinking of one layer. Then it becomes, to a reasonable first approximation, a two-dimensional problem such as finding the minimum size of a square that holds n^2 circles of radius r . What is the density of such an arrangement? This is better than any other rectangular arrangement when n is small, but eventually an arrangement more like B than A of question 10 comes to have a higher density than an arrangement within a square. Or does it? Investigate the smallest rectangular area into which to pack k circles all of radius r . Will the rectangle of smallest area that holds 7 circles in fact always hold 8?

Continue the previous investigation into three dimensions. What are different regular arrangements of spheres, and what are their densities? The problem goes back to Kepler and was first solved by Gauss. If you allow irregular packings, the problem is incredibly difficult and was finally solved only in 1998 by Thomas Hales with computer assistance. See George Szpiro's book, *Kepler's Conjecture*, for a popular account of this history.

A closely related problem is that of the so-called Kissing Number, that is, the largest number of spheres that can simultaneously touch a single sphere all of the same size. For circles in the plane the answer is 6. In three dimensions, it was the subject of a famous argument between Newton and Gregory, with a debate over whether the answer should be 12 or 13. An interesting physical experiment was done in the early 18th century. Dried peas were placed in a kettle with water and allowed to expand; the result was the peas were “formed into pretty regular Dodecahedrons” (Hales, 1731). This indicated that, perhaps, the answer should be 12, which is correct but wasn't proved until 1874!

References

Hales, S. (1731). *Statistical essays: Containing vegetable staticks* (Vol. 1). London: Printed for W. Innys, T. Woodward, and J. Peele.

Szpiro, G.G. (2003). *Kepler's conjecture: How some of the greatest minds in history helped solve one of the oldest math problems in the world*. Hoboken, NJ: Wiley.

FLIPPING FOR A GRADE

Teacher's Guide — Getting Started

Michael Cho
Bayside, NY

Purpose

In this two-day lesson, students play different coin-flipping games and try to understand what the outcomes may be. The objective of this lesson is to understand the meaning of expected value and standard deviation and why they are so important.

Prerequisites

Students should understand mean, median, mode, and range.

Materials

Required: Coins and internet access for research.

Suggested: None.

Optional: Graphing calculators.

Worksheet 1 Guide

The first four pages of the lesson constitute the first day's work. Students are introduced to two different coin-flipping games. Students become familiar with the rules and how the games work, and then determine the "typical" (expected value) outcome of each game. Students should be given some time and space to flip a coin and tally their points. If coins are not available, the students can be shown how to use a graphing calculator to generate random flips. The calculator function **randInt**(0,1) will randomly generate either a 0 or 1 which can be substituted for heads or tails and can be found under MATH PROB menu on a TI calculator.

Worksheet 2 Guide

The fifth and sixth pages of the lesson constitute the second day's work. Students continue to analyze and play with the coin-flipping games. Students try to determine the difference of the "swings" (standard deviations) of the two games and develop the meaning of standard deviation.

CCSSM Addressed

S-ID.2: Use statistics appropriate to the shape of the data distribution to compare center and spread of two or more different data sets.

S-MD.3: (+) Develop a probability distribution for a random variable defined for a sample space in which theoretical probabilities can be calculated; find the expected value.

S-MD.5: (+) Weigh the possible outcomes of a decision by assigning probabilities to payoff values and finding expected values.

S-MD.7: (+) Analyze decisions and strategies using probability concepts

FLIPPING FOR A GRADE

Student Name: _____ Date: _____

Your mathematics teacher has decided that instead of a test, you and your classmates will have the option of playing a game! Each student has the choice of picking one of two games, both of which involve flipping a coin ten times.



Source: U.S. Mint

GAME 1

Flip a coin ten times. For each head, the student wins 2 points, but for each tail, the student loses 1 point.

GAME 2

Flip a coin ten times. For each head, the student wins 100 points, but for each tail, the student loses 99 points.

Your grade depends on your final score:

- Lower than -10 and you will receive an F.
- Between -10 to 0 and you will receive a D.
- Between 1 to 10 and you will receive a C.
- Between 11 to 99 and you will receive a B.
- Higher than 100 and you will receive an A.

Leading Question

Which game should you pick in order to get the best grade?

FLIPPING FOR A GRADE

Student Name: _____ Date: _____

Analyze Game 1 first.

1. Assume that the coin is perfectly fair. Estimate the total number of points you believe someone will end up with if they play Game 1. Support your estimation.

2. Play Game 1! Flip a coin ten times and fill in the table below with your results. Sum your results in the bottom row.

Flip number	Heads or Tails	Points
1		
2		
3		
4		
5		
6		
7		
8		
9		
10		
TOTAL		

FLIPPING FOR A GRADE

Student Name: _____ Date: _____

Expected value is the weighted average of all the possible values. It is found by multiplying the probability of an event occurring by its expected outcome.

6. What is the expected value of points for Game 1? What is the expected value of points for Game 2? Compare these two values. What determinations about these two games can you make?

7. Find the expected value of flipping a fair coin 100 times where for each head, you win 2 points and for each tail, you win 1 point. Find the expected value of flipping the same unfair coin from the previous problem 100 times. Does the expected value match with your predictions in question 4 and 5? Why or why not?

FLIPPING FOR A GRADE

Student Name: _____ Date: _____

Recall the rules for Game 2. Flip a coin ten times. For each heads, win 100 points, but for each tails, lose 99 points.

8. Play Game 2! Flip a coin ten times and fill in the graph below with your results. Sum your results in the bottom row.

Flip number	Heads or Tails	Points
1		
2		
3		
4		
5		
6		
7		
8		
9		
10		
TOTAL		

Did you get the same number of points as when you played Game 1? If so, explain why, and if not, explain what was different with Game 2.

9. Can you create a model (such as a formula) that takes into account how much more “swing” or variation there is in Game 2 than Game 1? What variables should you incorporate?

FLIPPING FOR A GRADE

Teacher's Guide — Possible Solutions

The solutions shown represent only some possible solution methods. Please evaluate students' solution methods on the basis of mathematical validity.

1. Most students will answer that the number of points they think that they'll receive is 5. This can be obtained mathematically by calculating the expected value of one flip $[(0.5 \times 2) + (0.5 \times (-1))] = 0.5$ and multiplying the expected value of one flip by ten to obtain the expected value of ten flips, $0.5 \times 10 = 5$.
2. Answers will vary, but most games should give results that are close to the expected value.
3. Answers will vary, but generally, the estimates and results will be similar. Students might be surprised when the results are not "perfect" in that heads and tails did not each appear exactly half of the time.
4. Similar to question 1, the expected value will be $100 \times [(0.5 \times 2) + (0.5 \times (-1))] = 50$.
5. The expected value is $100 \times [(0.25 \times 2) + (0.75 \times (-1))] = -25$.
6. The expected value of Game 1 is $10 \times [(0.5 \times 2) + (0.5 \times (-1))] = 5$. The expected value of Game 2 is $10 \times [(0.5 \times 100) + (0.5 \times (-99))] = 5$. While the expected values of the two games are the same, students should point out that the Game 1 and Game 2 are still very different games. Good responses should have some mention of "swing" or variations.
7. The expected value of 100 flips of a fair coin is 50 and the expected value of 100 flips of the unfair coin described is -25.
8. Answers will vary. There will be a much larger variation in points in this game.
9. Answers will vary. Responses should include some mention of range, minimum, maximum, or quartiles.
10. Answers will vary. A possible solution for the formula of the mean of the mean is $(1/2)[(\text{maximum value} - \text{mean}) + (\text{mean} - \text{minimum value})]$.
11. The general formula for standard deviation for discrete random variables is

$$\sigma = \sqrt{\frac{1}{N} [(x_1 - \mu)^2 + (x_2 - \mu)^2 + \dots + (x_N - \mu)^2]} \text{ where } \mu = \frac{1}{N} (x_1 + x_2 + \dots + x_N). \text{ The formula for standard}$$

deviation takes into account the mean, the difference between each number and the mean, and the number of numbers.

12. Answers will vary. The modified formula should include more of the ideas that standard deviation includes.
13. Answers will vary. Most responses should be acceptable as long as they are mathematically accurate or valid.

FLIPPING FOR A GRADE

Teacher's Guide — Extending the Model

It might be interesting to see what the effect of the choice of game has on the actual grade.

Game 1: The chance of getting a grade of:

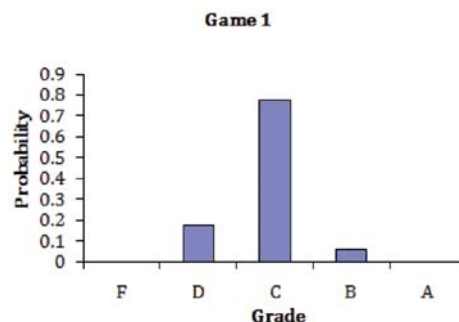
$$F = 0;$$

$$D = \left[\binom{10}{0} + \binom{10}{1} + \binom{10}{2} + \binom{10}{3} \right] / 2^{10} = 0.172;$$

$$C = \left[\binom{10}{4} + \binom{10}{5} + \binom{10}{6} + \binom{10}{7} \right] / 2^{10} = 0.773;$$

$$B = \left[\binom{10}{8} + \binom{10}{9} + \binom{10}{10} \right] / 2^{10} = 0.055; \text{ and}$$

$$A = 0.$$



This looks like a somewhat skewed but unimodal distribution which will be well described by a mean and a standard deviation. Note that it is impossible to get an A or to fail!

Game 2: The chance of getting a grade of:

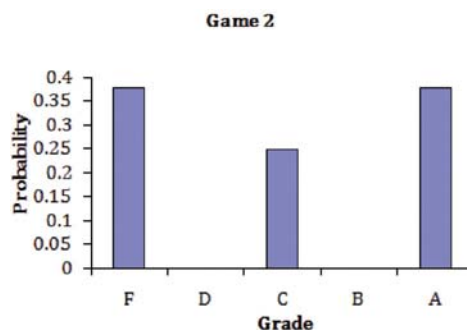
$$F = \left[\binom{10}{0} + \binom{10}{1} + \binom{10}{2} + \binom{10}{3} + \binom{10}{4} \right] / 2^{10} = 0.377;$$

$$D = 0;$$

$$C = \left[\binom{10}{5} \right] / 2^{10} = 0.246;$$

$$B = 0;$$

$$A = \left[\binom{10}{6} + \binom{10}{7} + \binom{10}{8} + \binom{10}{9} + \binom{10}{10} \right] / 2^{10} = 0.377$$



This looks like a symmetric but far from unimodal distribution! In fact, it is bimodal. The mean is all right, but a picture makes a far greater contribution to describing this distribution than a standard deviation. It is impossible to get a B or a D!

PRESCIENT GRADING

Teacher's Guide — Getting Started

Nazar Rabadi & Andrew Sanfratello
New Rochelle High School & Mohegan Lake, NY

Purpose

In this two-day lesson, students will learn how to approximate test grades given homework grades. They will construct a scatter plot and use the line of best fit to predict grades, as well as examine the effect the correlation coefficient and the residual have on the predictions.

Prerequisites

Students need to be able to create scatter plots and lines of best fit on their graphing calculators. Understanding of linear equations and awareness of some of their properties such as the y -intercept is necessary. Familiarity with correlation coefficient is helpful, although it is possible to introduce it in this lesson with supplemental work.

Materials

Required: Graphing calculators.

Suggested: Data sets of homework and test grades from a previously completed class.

Optional: None.

Worksheet 1 Guide

The first three pages of the lesson constitute the first day's work in which students are presented with homework and test grades for a class. It is suggested that the teacher provide real data for students to use, but if those are unavailable, fabricated data are provided. Students examine the data for patterns and consistencies, and then explain their ideas and support their claims using mathematics.

Worksheet 2 Guide

The fourth through sixth pages of the lesson constitute the second day's work in which students create a scatter plot to support their claims. They then calculate a line of best fit using graphing calculators to predict test scores from their data. Correlation coefficients are considered to help determine if the line of best fit is accurate. Once the line of best fit is used to predict scores, the residual is calculated to make sure that the predictions were reasonable.

CCSSM Addressed

S-ID.7: Interpret the slope (rate of change) and the intercept (constant term) of a linear model in the context of the data.

S-ID.8: Compute (using technology) and interpret the correlation coefficient of a linear fit.

S-ID.9: Distinguish between correlation and causation.

PRESCIENT GRADING

Student Name: _____ Date: _____

Teachers always are seeking ways predict the future grades of their students! Both teachers and students want grades to improve and to get high marks on tests and report cards.

Student	Quarter 1 Homework Grade	Quarter 1 Test Grade	Quarter 2 Homework Grade
Alison	91%	95%	88%
Chase	78%	70%	85%
Michelle	90%	83%	88%
Frank	70%	62%	75%
Junaita	84%	88%	90%
Cho	80%	76%	80%
Allen	77%	71%	81%
Yolanda	50%	59%	10%
Mary	79%	72%	70%
George	25%	41%	0%
Krystle	84%	81%	92%
Bailey	84%	77%	80%
Jordan	70%	66%	80%
Jamie	90%	80%	82%
Jenna	40%	49%	72%
Erick	100%	92%	97%
Tamara	95%	97%	92%
Ulysses	85%	79%	98%
Sal	47%	39%	84%
Jeremy	66%	58%	30%
Nicole	90%	87%	98%
Sean	80%	71%	85%
Warren	87%	81%	91%
Linda	65%	50%	79%
Rita	85%	77%	62%
Jessica	95%	91%	99%
Cristina	86%	90%	72%
Beth	88%	78%	90%
Larry	74%	65%	78%
Robert	91%	84%	88%
Quincy	83%	75%	88%
Darius	90%	84%	97%
Harrison	97%	93%	91%

Leading Question

Will students benefit from being required to do and hand in homework, how can teachers predict the future performances of their students, and in what ways can students improve their grades?

PRESCIENT GRADING

Student Name: _____ Date: _____

1. Collect the homework and test grades from a previously completed class. Focus on one grading period from the data and determine if there is any relationship between homework and test grades.

If no data are available, use the data on the first page.

2. What mathematics did you use to determine a relationship? If you used no mathematics, how could you use some to support your claim?

3. How might the relationship between homework grades and test grades differ if (a) the homework grade is calculated by what students turn in and is graded by the teacher, versus if (b) the homework grade is calculated by what students turn in, is graded, *and is returned to students with comments*.

PRESCIENT GRADING

Student Name: _____ Date: _____

4. How might the relationship between homework grades and test grades differ between the two grading schemes in question 3, versus if (c) the homework grade is calculated only by what is turned in.

If students just have to hand in something but the teacher does not look it over, how might that affect the quality of the work submitted?

5. What would you use to show a strong relationship or a weak relationship between homework grades and test grades?

6. With your data, apply the method you devised from question 5. Does your method have algebraic attributes? Does it have graphical attributes? Explain why you chose your method and how it might be applied to any random sampling. How might you use the data from one grading period to predict the test or homework grades of a subsequent grading period?

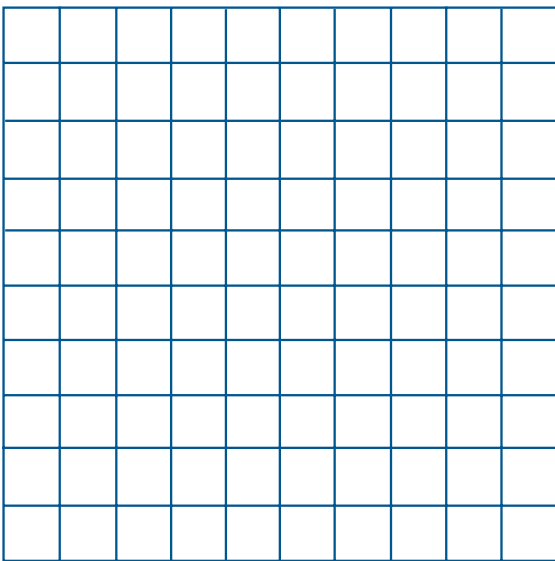
If your data are from a large set, take a small sample of that set that is easier to handle.

PRESCIENT GRADING

Student Name: _____ Date: _____

Use the data you collected from the previous class to answer the following questions. If no data were collected, use the data provided on the first page as well as the additional data provided.

7. Construct a scatter plot of the difference between the test grades from one grading period to the next and the difference in homework grades over the same two grading periods. Do you think that the homework grade is a good way to gauge what a student will score on a test? Does earning a good homework grade indicate anything in terms of test grade? Justify your answer. Do you think that earning good grades on homework caused an increase in the students' test grades? Explain your reasoning.



Student	Quarter 2 Test Grade
Alison	90%
Chase	81%
Michelle	80%
Frank	63%
Junaita	85%
Cho	84%
Allen	80%
Yolanda	29%
Mary	79%
George	33%
Krystle	85%
Bailey	73%
Jordan	71%
Jamie	79%
Jenna	61%
Erick	91%
Tamara	90%
Ulysses	93%
Sal	71%
Jeremy	44%
Nicole	93%
Sean	77%
Warren	89%
Linda	84%
Rita	70%
Jessica	100%
Cristina	78%
Beth	92%
Larry	72%
Robert	90%
Quincy	83%
Darius	95%
Harrison	89%

8. Using technology, find the line of best fit for the difference in test grades and homework grades. Interpret the y -intercept and the slope of the line in the context of the problem.

*What is a line of best fit?
What properties should it
have?*

PRESCIENT GRADING

Student Name: _____ Date: _____

9. Using technology, find the **correlation coefficient**, r , and describe what this means.

What is a correlation coefficient? Research it!

10. Using the line of best fit calculated in question 8 and homework grades from the next grading period, predict the test scores for the next grading period. What do you notice about the predictions? Do they make sense? What other variables might you use to create a function for calculating test scores?

Student	Quarter 3 Homework Grade
Alison	75%
Chase	90%
Michelle	92%
Frank	77%
Junaita	88%
Cho	76%
Allen	89%
Yolanda	23%
Mary	62%
George	75%
Krystle	71%
Bailey	69%
Jordan	97%
Jamie	98%
Jenna	88%
Erick	98%
Tamara	93%
Ulysses	91%
Sal	77%
Jeremy	35%
Nicole	82%
Sean	83%
Warren	84%
Linda	80%
Rita	61%
Jessica	96%
Cristina	66%
Beth	88%
Larry	75%
Robert	93%
Quincy	90%
Darius	89%
Harrison	83%

PRESCIENT GRADING

Student Name: _____ Date: _____

11. The residual is the actual value minus the predicted value. Calculate the residuals for your results above. Do you think it is better to have the absolute value of the residual be small and close to 0 or should it be greater? Explain your reasoning.

12. Discuss the students' actual test grades and their predicted test grades. Now that all the test scores are in, based on the line of best fit determined earlier, what can you do to get a better prediction? Do you think the line of best fit is a good one?

13. What kind of an effect, if any, do you think including a new student's homework and test scores would have on the line of best fit? In what other situations might this method for prediction be used? Research if you cannot think of any.

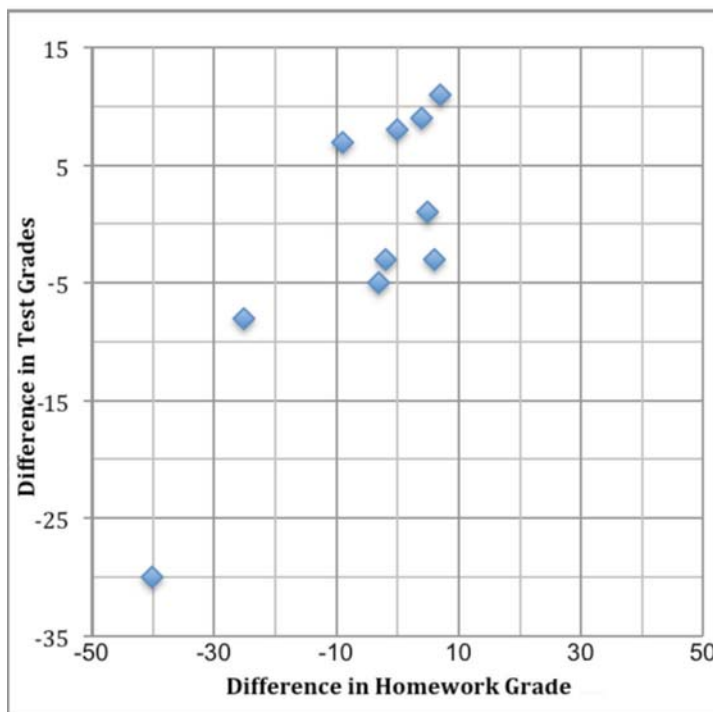
Student	Quarter 3 Test Grade
Alison	85%
Chase	92%
Michelle	93%
Frank	81%
Junaita	85%
Cho	71%
Allen	80%
Yolanda	45%
Mary	69%
George	62%
Krystle	70%
Bailey	77%
Jordan	92%
Jamie	88%
Jenna	81%
Erick	100%
Tamara	89%
Ulysses	82%
Sal	71%
Jeremy	10%
Nicole	72%
Sean	74%
Warren	83%
Linda	70%
Rita	57%
Jessica	91%
Cristina	78%
Beth	91%
Larry	79%
Robert	95%
Quincy	88%
Darius	90%
Harrison	77%

PRESCIENT GRADING

Teacher's Guide — Possible Solutions

The solutions shown represent only some possible solution methods. Please evaluate students' solution methods on the basis of mathematical validity.

1. From the artifact data, there is a strong relationship between the homework grade and the test grade from the first quarter. If using other data, answers will vary.
2. Scatter plots, lines of best fit, and correlation coefficients are some of the mathematical tools that can be used to determine the statistical relationships between the scores.
3. If the homework assignments are returned to students with comments, it is expected that students would learn from their mistakes and perform better on tests.
4. If the homework assignments are only required to be turned in, it is expected that students would not perform as well as in cases (a) or (b) from question 3.
5. If students used the line of best fit, then if r is closer to 1, a strong positive linear relationship exists between homework and test grades. If r is closer to 0, a weak positive linear relationship exists between homework and test grades.
6. Scatter plots are one way to represent the data graphically. Lines of best fit are ways to represent the data algebraically. Explanations will vary.
7. A scatter plot is pictured to the side using the data from the first ten students. This data shows a strong positive relationship between homework score and test score. Answers for different data will vary.
8. The line of best fit for the first ten students is $y = 0.6528x - 2.4207$. Earning the same homework grade in the second quarter, a student would expect to earn about 2.5 points fewer on the second quarter test. For each point increase in homework scores, a student would expect to earn an increase of about 0.65 points on their test. Answers for different data will vary.
9. $r = 0.8316$. Since r is near 1, there is a strong positive, linear relationship between the increase in homework grades and the increase in test grades. Answers for different data will vary.
10. The predicted grades for the first ten students are in the table on the following page. Answers for different data will vary.
11. The residuals for the first ten students are in the table provided. Answers for different data will vary. It is better to have the absolute value of the residual close to 0, as a small residual indicates a more accurate predicted test score.



PRESCIENT GRADING

Teacher's Guide — Possible Solutions

12. Students whose test scores provided residuals with a greater absolute value affected the line of best fit the most. One option is to eliminate these students to create a more accurate line of best fit.
13. If the new student performs near their predicted score, the absolute value of their residual will be small, and the sample will provide a more accurate line of best fit.

Student	Predicted Test Grade Quarter 3	Quarter 3 Test Residuals
Alison	79.1%	5.9
Chase	81.8%	10.2
Michelle	80.2%	12.8
Frank	61.9%	19.1
Juanita	81.3%	3.7
Cho	79.0%	-8
Allen	82.8%	-2.8
Yolanda	35.1%	10
Mary	71.4%	-2.4
George	79.5%	-18

PRESCIENT GRADING

Teacher's Guide — Extending the Model

This particular module contains six tables with thirty-three entries each. They represent homework and test grades for each student in a class. A question that is not part of the usual initial experiences in data analysis, but might be interesting, is whether these numbers look like they are taken from a real class or were made up to resemble the kinds of sequences real tests and homework might lead to. Real grades tend to have certain regularities, partly from the nature of grading and partly from the accidental patterns that tend to occur in most sets of numbers with a large random component. Of course it's hard to be sure, but you can look for features that seem a bit unusual. For example, the table of quarter 1 homework grades contains seven numbers that appear more than once: 70, 80, 84, 85, 90, 91, and 95. Together, they represent half the data! The fact that there *are* repetitions is quite realistic. Should there be that many? Notice the quarter 2 test grades have very few repetitions. On the other hand, something that stands out in the quarter 2 test grades is that there are 7 multiples of 10, and 2 other multiples of 5. Fine, but of the remaining numbers, 6 are even, and 18 are odd! Isn't that a bit strange? Would grading tests lead to that kind of result? There is no certainty in any of this, but it's interesting to have a look.

The question of the likelihood of an appearance of a random sequence deserves examination. The simplest experiment of this nature of which I am aware is to ask students to produce a random sequence of n 0s and 1s, or similarly, heads and tails, or, with respect to question of the grades, evens or odds. Each student can do it one of two ways: the first way is to use a table of random numbers or actually toss a coin, and the second way is to make it up oneself to look random. The professor claims that she will be able to tell just by looking at a sequence which method the student used! She can do this because she knows the key fact, which is that a truly random sequence will have a fair number of accidental regularities that an amateur trying to make up a random-looking sequence would probably not allow.

Let's consider one particularly simple phenomenon in a sequence of n 0s and 1s: Is there a string of 3 or more consecutive identical digits, that is, 3 0s or 3 1s? Some students might feel intuitively that such a string is a little too much regularity for a random sequence. In fact, by the time n is 13, fewer than 10% of all binary strings of length 13 have no triple of either all 0s or all 1s consecutively, and that proportion is down to 1% by the time n is in the low 20s.

How do you compute something like that? What we want to find is a recursion of A_n , which we define as the number of n -bit binary strings that DO NOT contain a triple of either all 0s or all 1s in immediate succession. It is useful to check that $A_3 = 6$, $A_4 = 10$, and $A_5 = 16$. To get a recursion relation, we consider the set of n -bit strings containing no triples of any kind to be made up of four disjoint subsets, a_n , b_n , c_n , and d_n . Define a_n , b_n , c_n , and d_n to be the number of n -bit strings that contain neither 3 consecutive 0s nor 3 consecutive 1s AND whose last 2 bits are 00, 01, 10, and 11, respectively. So, for example, $a_4 = 2$, $b_4 = 3$, $c_4 = 3$, and $d_4 = 2$, which add up to $A_4 = 10$, as they should. Now imagine that you have all the strings that you counted in A_n sorted into their last two bits, and now add one more bit at the end of each string. Of course you must avoid making any forbidden triples – you cannot add a 0 to a string ending in 00, or a 1 to a string ending in 11. You will find that a_n can create only a b_{n+1} , because you can only adjoin a 1 to a terminal 00. On the other hand, b_n can create both a c_{n+1} and a d_{n+1} ; c_n can create both an a_{n+1} and a b_{n+1} , and d_n can create only a c_{n+1} . We get four equations: $a_{n+1} = c_n$, $b_{n+1} = a_n + c_n$, $c_{n+1} = b_n + d_n$, and $d_{n+1} = b_n$. This tells you that $A_{n+1} = A_n + b_n + c_n$. But we also have that $b_n + c_n = A_{n-1}$. Hence we get a three-term recursion relation, namely just $A_{n+1} = A_n + A_{n-1}$.

PRESCIENT GRADING

Teacher's Guide — Extending the Model

But that is the recursion for the Fibonacci numbers! We see, in fact, that A_n is equal to $2F_{n+1}$! You may have recognized the (6, 10, 16) we had earlier as $2(3, 5, 8)$. That's how we got some of the percentages of all binary strings that we mentioned before. We simply had to find $2F_{n+1}$, the number of n -bit strings WITHOUT 3 consecutive 0s or 1s, and divide by 2^n , the number of all n -bit strings. Notice the denominator of this ratio grows much more quickly than the numerator. What we have seen is that for n -bit strings of length more than the low teens, the unusual phenomenon will be to have NO triples of the same digit. So the moral of the story is if you want your n -strings to look random, don't avoid the "accidental" consecutive appearances of the same digit!

PICKING A PAINTING

Benjamin Dickman
Brookline, MA

Teacher's Guide – Getting Started

Purpose

In this two-day lesson, students are asked to choose the best possible painting from a group provided to them. Certain restrictions prevent students from going back to previously viewed paintings, so choosing the best is not as straightforward as just looking at all of them and deciding.

The objective of this lesson is to use ordering and logical thinking to create probabilistic strategies that have greater chances of success than just random selection. Conditional probability is also explored as a way to evaluate the strategies further.

Prerequisites

Knowledge of factorials is helpful but not necessary.

Materials

Required: None.

Suggested: None.

Optional: Playing cards to represent paintings of greater and lesser value.

Worksheet 1 Guide

The first three pages of the lesson constitute the first day's work. Students are introduced to the problem of choosing a painting for an art gallery. Shrinking the problem down to a situation in which there are only two or three paintings helps students to create a strategy for picking the best painting possible. The idea of conditional probability is introduced near the end of the first day.

Worksheet 2 Guide

The fourth page of the lesson constitutes the second day's work. Students are urged to create a general formula for conditional probability and then expand their process of picking a painting to larger selections of paintings.

CCSSM Addressed

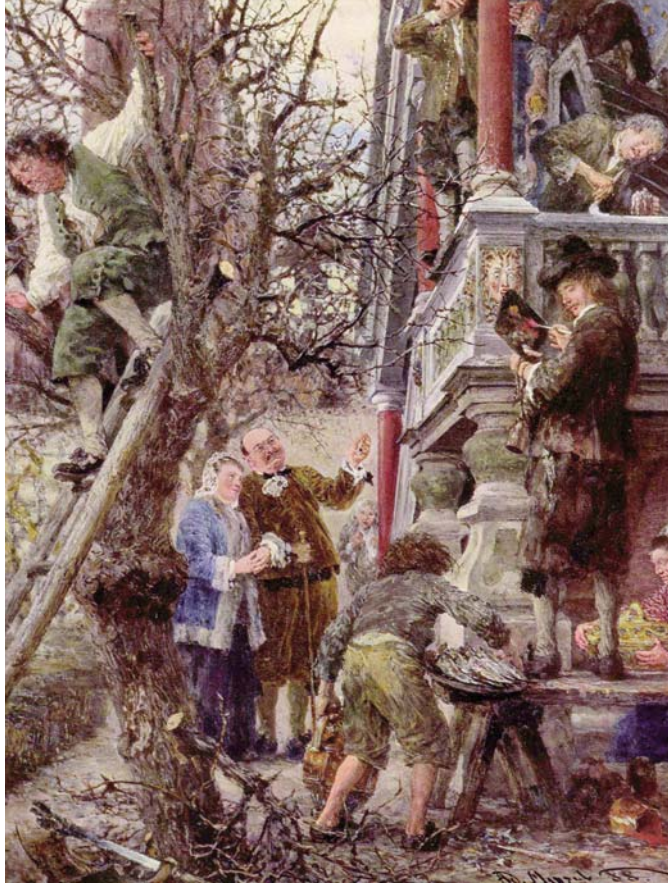
S-CP.1: Describe events as subsets of a sample space (the set of outcomes) using characteristics (or categories) of the outcomes, or as unions, intersections, or complements of other events ("or," "and," "not").

S-CP.3: Understand the conditional probability of A given B as $P(A \text{ and } B)/P(B)$, and interpret independence of A and B as saying that the conditional probability of A given B is the same as the probability of A , and the conditional probability of B given A is the same as the probability of B .

PICKING A PAINTING

Student Name: _____ Date: _____

An anonymous donor has decided to give her art collection to various museums. Each museum is allowed to choose one painting, and, because you have a discerning eye for brushwork, the National Gallery of Art has requested that you choose on their behalf. Furthermore, because the paintings all are different, you are confident that no two of them are “equally good.”



Adolph von Menzel [Public domain], via Wikimedia Commons

Time constraints and other museums vying for the paintings force you to follow a few rules:

1. You cannot view any painting before it is shown officially;
2. Paintings will be shown one at a time, in a random order;
3. For each painting, you must either choose it or reject it;
4. If you choose a painting, you must leave with it;
5. If you reject a painting, you cannot return to it later;
6. The total number of paintings is known ahead of time; and
7. You know the relative rankings of paintings that were shown and have no external knowledge.

Leading Question

How will you decide which painting to choose if your goal is to pick the best painting possible?

PICKING A PAINTING

Student Name: _____ Date: _____

1. Is it a good idea always to pick the first painting shown? What about the last one? What other strategies could you use?
2. Suppose there are only two paintings. What is the chance that the first painting shown is the best one? What is the chance that the last painting shown is the best?
3. What if there are three paintings? What is the chance that the first painting shown is the best? What is the chance the second one shown is best? What is the chance the third one shown is best?
4. For three paintings, there will be the best painting (A), the second best painting (B), and the worst painting (C). What are the different orderings in which the three paintings could be shown? How many of these orderings are there in all?

*The set of all possible outcomes is known as the **sample space**.*

PICKING A PAINTING

Student Name: _____ Date: _____

9. How many orderings are possible for five paintings? Six? Create and evaluate strategies for when there are many paintings. What difficulties might emerge?

10. Create a general formula for calculating the probability if you know the quality of the first painting shown.

11. What if there were 100 paintings? Create a strategy that will help you pick the best painting at least $1/4$ of the time

Try first dividing the paintings into two equal sets, one of the first 50 paintings shown, and the other containing the last 50 paintings shown.

12. How might you generalize the question of choosing the best painting? What are some related questions you can ask?

PICKING A PAINTING

Teacher's Guide — Possible Solutions

The solutions shown represent only some possible solution methods. Please evaluate students' solution methods on the basis of mathematical validity.

1. Always picking the first or last painting will result in the same probability of picking the best painting. With n paintings, there will be probability $1/n$ of picking the best.
2. Using the same idea as question 1 where $n = 2$, $P(\text{first shown is the best}) = P(\text{last shown is the best}) = 1/2$.
3. Similar to the last two, $P(\text{first shown is the best}) = P(\text{second shown is the best}) = P(\text{last shown is the best}) = 1/3$.
4. Six orderings create the sample space: (A, B, C), (A, C, B), (B, A, C), (B, C, A), (C, A, B), and (C, B, A).
5. The strategy suggested by a friend is successful for the subset of the sample space (C, A, B), (B, A, C), and (B, C, A), and fails for the rest. Since it is successful for 3 of the 6 orderings, the probability of the strategy being successful is $P(\text{strategy is successful}) = 3/6 = 1/2$.
6. When B is shown first, the best, A, is chosen $2/2$ times. When A is shown first, the best is chosen $0/2$ times, and when C is shown first the best is chosen $1/2$ times.
7. For four paintings, there are $4! = 24$ orderings possible. The same strategy as before (reject the first painting, and pick the next one shown that is better than the first) will be successful with probability $11/24$. Another option is to choose to reject the first two paintings, then choose the next one shown that is better than both of the first two. This will be successful with probability $10/24 = 5/12$.
8. If the paintings are ordered A, B, C, and D as in question 4, then if D is shown first, the best, A, is chosen $2/6$ times. When C is shown first the best is chosen $3/6$ times. When B is shown first the best is chosen $6/6$ times and when A is shown first the best is chosen $0/6$ times. This total aligns with the answer to question 7, $11/24$.
9. For n paintings, there are $n!$ possible orderings. Thus, for 5 paintings, there are $5! = 120$ orderings and for 6 there are $6! = 720$ orderings. Using similar strategies as with the previous problems, students may choose to view some number, k , of paintings before deciding when to stop viewing and choose a painting that is better than any of the ones already viewed. As n grows, computations may become very tedious very quickly.
10. The formula should bear some resemblance to conditional probability (i.e., given two events, X and Y , then the probability of X occurring given that Y has occurred is $P(X \text{ given } Y) = P(X \text{ and } Y)/P(Y)$).
11. Reject any painting shown in the first half, and choose the next painting shown that is better than any of those shown in first half. This strategy will succeed at least when the second best painting is in the first half, and the best painting is in the second half. The probability of this is $(50/100)(50/99) > 1/4$. (In fact, there are other cases for which this strategy will work that will only increase the probability that it is successful.)
12. Choosing the best painting is not always possible no matter what strategy is used. The best thing do is to increase the probability of choosing one of the best paintings (if not the best). Some possible questions are "For a total of n paintings, how many should you pass on?", "Are there other *kinds* of strategies one could use?", and "What are the advantages and disadvantages of using a computer program to evaluate probabilities of success for different strategies?"

PICKING A PAINTING

Teacher's Guide — Extending the Model

The first extension is to define the optimal strategy for n paintings, and to do most of the proof that it is correct.

First, a definition we will need: A **candidate** is a painting which you rank higher than any you have seen *previously*.

We begin from the fact that the best painting among the n paintings is somewhere in the order in which the collection is shown. The strategy we will consider, which generalizes one in the lesson, is to examine and to rank relatively the first $p-1$ of the paintings shown. What is p ? We will show how to find the best p as a function of n . Then the strategy is to accept the first painting which is a candidate shown after the initial $p-1$ paintings. This means the first painting you see starting at p that is better than all of the first $p-1$ paintings is the one you choose. You are in a sense using the first $p-1$ paintings to “get the lay of the land”.

What will this strategy do? If the very best of all the paintings happens to be among the first $p-1$ that you looked at, you have missed it and there is nothing you can do about it. In this case, your probability of getting the best painting is 0. This consideration will keep p from getting too large.

If the best painting is later than $p-1$ in the order of presentation, that is, in the interval (p, \dots, n) , you have a chance. If the painting presented as p happens to be the overall best, which happens with probability $1/n$, you will get it. Of course, $1/n$ is the probability of the best painting being at *any* particular position in the order. Suppose the overall best painting is at position k , where $k \geq p$. Will you get that painting? If and only if the best painting among the first $k-1$ paintings is in fact among the first $p-1$ paintings! What's the probability of that? The probability is just $(p-1)/(k-1)$. Thus $(p-1)/(k-1)$ is the probability that you will get the best painting *given that it is at position k in the order*. But the probability of that condition, as we have seen, is just $1/n$. Hence the probability that you will get the best painting when it is at position k is $(p-1)/(n(k-1))$. Hence the probability that you get the best overall painting is the sum of these expressions from $k = p$ to $k = n$. We write it out:

The probability of success for this strategy is

$$\sum_{k=p}^n \frac{1}{n} \frac{p-1}{k-1} = \frac{p-1}{n} \left(\frac{1}{p-1} + \frac{1}{p} + \frac{1}{p+1} + \dots + \frac{1}{n-1} \right), \quad 1 < p \leq n.$$

It remains to find the best p as a function of n . For $n = 3$ and 4, this was done in the lesson, and it is worth checking that the above formula gives the best answer. When you vary p with fixed n , the expression begins small when $p = 2$, is again small when $p = n$, and has a maximum in between. You look for the value of p where it stops increasing and starts decreasing. Let's also look at this for large n . About how big is that sum we just defined? For large n and p , it is approximately $(p/n)(\ln n - \ln p)$. If we set $x = n/p$, this is $(\ln x)/x$. This has a maximum at $x = e$. So the best strategy, if n is at all large, is to pick p/n as close as we can to $1/e$.

We said at the beginning that we will do “most of the proof” that this is correct. We have omitted the argument that the optimal strategy is, in fact, to look at some number $p-1$ of paintings and then pick the best thereafter. This is eminently reasonable, but that's not a proof. The full story can be found, for example, in Fred Mosteller's *Fifty Challenging Problems in Probability with Solutions*.

And now, a second extension: An interesting modeling problem in a very similar spirit is what is sometimes called the “theater problem”. It concerns finding a parking space when you want to go to the theater. Imagine an infinite road with parking spaces at the integers, most of which are filled as you approach the theater, which is at a known integer location. The model is actually most workable if you assume an infinite road.

PICKING A PAINTING

Teacher's Guide — Extending the Model

You want a space as close to the theater as possible. When you consider a candidate, that is, a space that is available, you cannot tell what other closer spaces might be available. If you don't take this candidate, the space will no longer be there if you later decide you should have taken it. If you don't take a space by the time you pass the theater, you will have to take one a long way beyond the theater, and you will be unhappy.

Assuming you know the location of the theater, and the probability that any space will be available, what is your best strategy? Once you understand this one, you can consider including in the model a (possibly high) cost of making an illegal U-turn and trying again!

Reference

Mosteller, F. (1987). *Fifty challenging problems in probability with solutions*. New York: Dover.

CHANGING IT UP

Teacher's Guide — Getting Started

Heather Gould
Stone Ridge, NY

Purpose

In this two-day lesson, students will examine the United States monetary system and make mathematical judgments about how to stock a cash register till (the drawer containing the money that “pops out” of the register). Different situations are modeled, each time refining the initial model.

Introduce the students to the situation to be modeled: a cash register till needs to be stocked with extra coin rolls. Cashiers want to try to run out of all the types of change at about the same time so they need to cash in for new change as rarely as possible. Under-stocking doesn't work because running out of coins too frequently results in longer waiting times for customers, and supervisors have to supply more change for the cashier. The till cannot be overstocked with coin rolls because it will be too heavy and will be very slow to open.

Prerequisites

Students should have a good understanding of algebra and averages. Familiarity with US currency is required.

Materials

Required: Internet access (for research).

Suggested: US currency manipulatives, a random sample of receipts.

Optional: Spreadsheet software.

Worksheet 1 Guide

The first three pages of the lesson constitute the first day's work where students determine what is meant by a “typical” (average) amount of each coin that is handed back in a cash transaction. Students first create a model of the situation and then refine it for greater accuracy.

Worksheet 2 Guide

The fourth and fifth pages of the lesson constitute the second day's work where students continue to work with their model, making further revisions when considering new information. A short time is spent investigating foreign currencies, and then the data collected is applied to their original model. Expected value is defined and students are instructed to use expected values to model the coin roll problems and compare it to their method. If the student initially used expected value in the model, they will try to create another model and compare it to expected value.

CCSSM Addressed

S-MD.2: (+) Calculate the expected value of a random variable; interpret it as the mean of the probability distribution.

S-MD.7: (+) Analyze decisions and strategies using probability concepts (e.g., product testing, medical testing, pulling a hockey goalie at the end of a game).

CHANGING IT UP

Student Name: _____ Date: _____

Most cash register tills (drawers) have a space for the cashier to store extra rolls of coins in case they run out of loose coins. Cashiers like to run out of extra rolls at the same time so they need to restock their change as infrequently as possible. They also know that having too few extra rolls will slow them down and irritate both the customers and supervisors; having too many will weigh down the drawer and make it difficult to open.



Photo © Comap, Inc.

Leading Question

How many rolls of each type of coin should be stocked in the cash register till?

CHANGING IT UP

Student Name: _____ Date: _____

4. Is your answer from question 3 reasonable? Are there other considerations that you left out that would make it more reasonable? What are they and how do they affect your answer?

5. Seasoned cashiers know that many customers who like to pay in cash also pay enough pennies so they only get “silver” back. How might this affect the typical number of coins you would give back in each transaction? How might it affect how many extra rolls you would stock in the till?

What is happening to the number of pennies that you get and give back? How about the other coins?

6. Retail stores often make set “change orders” from banks in order to stock up for the week ahead. If you were a manager of such a store, how would you determine the number of each type of coin roll to pre-order?

CHANGING IT UP

Student Name: _____ Date: _____

7. If you worked in an “All for 99¢” store and you knew that most customers only pay in cash for transactions that are less than \$20 when tax is included, would this affect how you would stock your extra rolls? If so, how many would you stock and why? If not, why wouldn't it affect it?

Determine the sales tax where you live. How does this affect your decision?

8. Suppose you still have the same situation as in question 7. After working there for some time, you notice that transactions whose final totals are less than \$10 happen about twice as often as transactions whose final totals are between \$10 and \$20. Might this affect how you would stock your extra coin rolls? Explain your reasoning and if it changed, how you would stock extra rolls now?

CHANGING IT UP

Student Name: _____ Date: _____

9. Many other countries use different denominations for each of their coins than in the US. Research one such country's monetary system and determine the best way to stock extra rolls of coins in their cash tills. Make sure to describe the different denominations of coins and why that affects your answer.

10. The **expected value** of a random variable is a "weighted average" of all the possible values the random variable can take on and each of their probabilities of occurrence. When all values are equally likely, the expected value is the same as the arithmetic mean. Did you use expected values in your solutions to the extra coin roll problems? If so, try to think of another method to solve questions 3 and 8; if not, use expected value to solve them. Which approach did you prefer? Which one seems best? Why?

CHANGING IT UP

Teacher's Guide — Possible Solutions

The solutions shown represent only some possible solution methods. Please evaluate students' solution methods on the basis of mathematical validity.

1. The most important thing to know and collect is how many of each type of coin are handed back in each possible transaction. Customers will receive 0 to 4 pennies, 0 or 1 nickel, 0 to 2 dimes, and 0 to 3 quarters. Students should note that the frequencies of these possibilities are not evenly distributed, and thus 1 dime, for example, is not the “average” number of dimes given back.
2. Assume all values of change are equally likely. Then there are 100 possible amounts of change to be given. In these 100 transactions, a total of 200 pennies (P), 150 quarters (Q), 80 dimes (D), and 40 nickels (N) are given back. This leads to $E(P)=2$, $E(Q)=1.5$, $E(D)=0.8$, and $E(N)=0.4$.
3. Only whole rolls can be stocked. Using $E(N)$ as a guide and rounding $E(Q)$ to 1.6, we see that we will need 1 roll of nickels, 2 rolls of dimes, 4 rolls of quarters, and 5 rolls of pennies. This is because $E(D)=2E(N)$, $E(P)=5E(N)$, and $E(Q)\approx 4E(N)$.
4. Most cashiers know that the answer to question 3 is unreasonable, although this will not be evident to students who have never worked with a cash register. There are simply too many rolls; most tills will not hold such a large number of extra rolls. It would have been more reasonable to use $E(D)$ to compare expected values. This will result in 1 roll of nickels (round 0.5 up), 1 roll of dimes, 2 rolls of quarters, and 2-3 rolls of pennies. This is much more reasonable, but still atypically large. Students might also consider that customers who pay in cash also tend to *give* the cashier coins, thereby changing the expected values of coins handed back.
5. Fewer pennies will be handed back *and* more will be received, so the number of penny rolls may be slightly reduced. More “silver” (nickels, dimes, and quarters) will be handed back, so more may need to be stocked.
6. Multiply each expected value by 10 to get whole numbers; multiples of 4 rolls of nickels, 8 rolls of dimes, 15 rolls of quarters, and 20 rolls of pennies should be ordered. Some students may further consider that there are 40 coins in a standard roll of nickels or quarters (\$2 and \$10, respectively) while dimes and pennies have 50 coins in a standard roll (\$5 and \$0.50, respectively).
7. Answers will vary based on the tax rate of the area. The model is severely affected by the restriction that there are at most 20 possible different values of change to be handed back. This changes all of the values assumed in question 3.
8. Answers will vary for the same reasons as in question 7. Regardless of the actual values, students may choose to “double-count” change values for transactions less than \$10 and consider values between \$10 and \$20 only once. “Typical” values can be calculated this way.
9. Answers will vary depending on the country chosen. In the United Kingdom, for example, no amount of change requires giving back more than 2 of any type of coin. (Values are 1p, 2p, 5p, 10p, 20p, 50p, £1, and £2, where 1p = £0.01.)
10. Answers will vary; students should show appreciation for the utility of expected value.

CHANGING IT UP

Teacher's Guide — Extending the Model

In light of your problem, would you favor the reintroduction of 50 cent pieces?

The five front compartments of the till are equal in width as well as in depth. This is not the best for the coins, but the way the tills are made, the width will be same front and back, and the widths for the bills have to be the same. It would be nice if you could have more space for the nickels because their size is disproportionate to their importance. You never see \$2 bills any more, so the fifth compartment often goes unused or for larger bills like fifties and hundreds. If you could reinvent the till, what changes would you make to accommodate extra rolls of coins and only four slots for the bills?

You often spend exactly \$3.30 for your favorite lunch. The regular cafeteria cashier, in giving you change, usually doesn't hand back 2 dimes and 2 quarters, she gives you 3 dimes, 3 nickels, and one quarter. To her, the nickels are a nuisance, fill up their compartment too quickly, and she doesn't want to run out of quarters! An interesting extension might therefore be to decide at what compositions of the till it would be to the cashier's advantage to hold on to more quarters and, when there is a choice, use up nickels at the rate of $1Q = 3N + 1D$.

Managing the boxes where the coins accumulated in coin phones used to be a really important problem. A coin-operated phone was designed to quit working when the coin box was full. You don't want that to happen, so you schedule emptying of coin boxes. Of course that's a statistical phenomenon. For x dollars you could install a prong sticking into the coin box somewhere near the top so that when the coin level reaches that prong, it sends an alerting signal to the central office so that they can send somebody to collect the coins. What's the optimal height for the prong? For what x is that worth it the expenditure? Would it be cheaper to put a second coin phone next to the first one so that the two boxes would fill more slowly? Would it be even better to have a public campaign asking people to use more dimes and quarters and fewer nickels in the coin phones, so they wouldn't fill up so fast? There's the problem with the volume of a nickel again.

Going farther afield: when the government first decided to replace silver dimes and quarters with laminated coins made of cheaper metal they turned to both Bell Labs and the slot machine industry in Nevada, because of the tests that a coin must pass through when it is used in a coin slot. A coin is tested for size, weight, and electrical conductivity, among other things. The "sandwich" had to pass the same tests that the old coins did — so what should the composition be?

At Halloween back in 1963, kids carried little cans to collect money for UNICEF as they went trick-or-treating. One church's UNICEF penny collection — 2,642 pennies in all — was used as a huge random sample to estimate the half-life of a penny. Lincoln Head pennies began to be minted in 1909, but the quantity didn't amount to anything until after 1930. Divide the number of pennies you have for any given year by the number minted in that year. Plot this ratio for every year, in our case from 1963 back to about 1930. Use a logarithmic scale for the ordinate only, keep a linear scale for the years. Then fit a line to the data, and you find a half-life of about 12 years. Called a log-linear plot, the slope gives you the exponent in the rate of decay. In a cash register, it would probably take you a long time to get enough pennies for a decent sample. Nowadays, dimes might be better for estimating the half-life of a coin. There is a natural historical cutoff when the silver dimes went out of circulation and were replaced by the Roosevelt laminated coins.

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